MATHEMATICS

5 points:

How many pairs (x,y) of natural numbers satisfy the equation $5x + 13y = 2015$?

Answer: 30

Solution:

$5x + 13y = 5 * 13 * 31$

We see that $x = 13m$ and $y = 5k$ so that $m,\ k$ are natural numbers and

$m + k = 31$

There are 30 different pairs (m, k) of natural numbers satisfying this equation, i.e.,

$(1, 30),\ (2, 29),\ \ldots\ (30, 1).$

10 points:

Find all pairs (x,y) of natural numbers satisfying the equation

$$\sqrt{x - \frac{1}{5}} - \sqrt{y - \frac{1}{5}} = \sqrt{5}\ .$$

Answer: The pairs

$(x, y) = \left(\frac{(n+5)^2+1}{5}, \frac{n^2+1}{5}\right), \quad n = \textit{natural number ending in } 2, 3, 7, 8$

Solution:

Let us solve the equation with respect to x

$\sqrt{x-\frac{1}{5}} = \sqrt{y-\frac{1}{5}} + \sqrt{5}$

$x - \frac{1}{5} = \left(\sqrt{y-\frac{1}{5}} + \sqrt{5}\right)^2$

$x = y + 5 + 2\sqrt{5y-1}$ (*)

We conclude that $\sqrt{5y-1} = n$ is an integer number (it cannot be half-integer as its square must be an integer). Then

$y = \frac{n^2+1}{5}$

and from (*)

$x = \frac{n^2+1}{5} + 5 + 2n = \frac{(n+5)^2+1}{5}$

As y and x are natural numbers, the number n^2+1 must be divisible by 5 which is possible only if n is a natural number ending in 2,3,7,8. For example,

$n = 2, 3, 7, 8, 12, 13, ...$ and

$(x, y) = (10, 1),\ (13, 3),\ (29, 10), ...$

PHYSICS

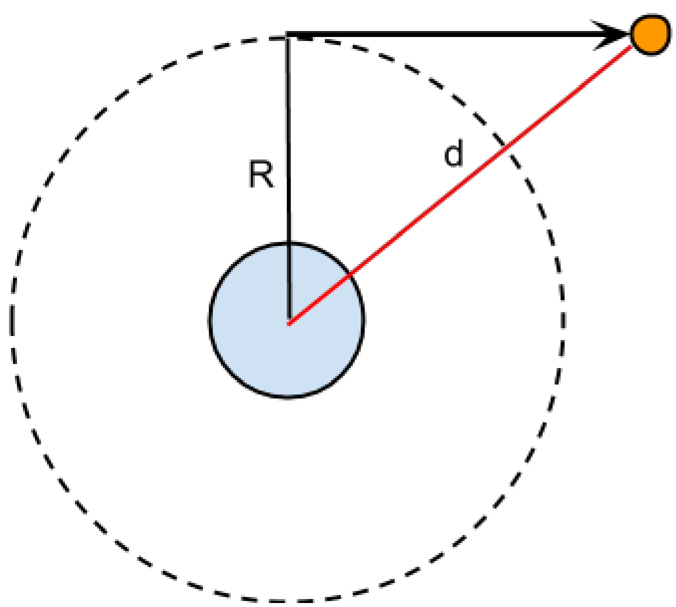
5 points:

Distance from the Earth to the Moon is about R=400,000km. The Moon makes a complete turn around Earth in 28 days.

Imagine that on the New Year Eve the gravity in the Universe is turned off. What will be the distance between the Earth and the Moon one week after that?

Solution.

Once the gravity is turned off, the Moon will leave the circular orbit and just continue moving along a straight line with the same speed as it had on the orbit. This means that in 1 week (¼ of lunar month) it will cover distance $2\pi R/4$. Initially it was at distance R from the Earth (see picture), so one can find the distance after one week by using the Pythagorean theorem:



$d = \sqrt{(\pi R/2)^2 + R^2} = R\sqrt{\pi^2/4 + 1} \approx 1.86R$

That's about 750,000km.

10 points:

A ball of radius *R*, mass *M*, and moment of inertia $I = \frac{2}{5}MR^2$, is spun with an angular velocity ϖ, and is put on a flat horizontal surface (that has friction). The ball first slides but eventually starts to roll without sliding. Find the speed of the ball once it rolls.

Solution.

When the ball contacts the surface, it first moves with constant acceleration, until the center-of-mass (COM) velocity becomes such that the condition of "no sliding" is satisfied. That is, until the translational velocity v, and the angular velocity ϖ of ball's rotation are related as $v = \varpi R$. The force providing the acceleration of the ball is the kinetic friction. Let its value be F, and the time it takes to reach the "no sliding" condition be t. By using the 2nd Newton's law we can find the accelaration $a = F/m$ and the final speed of the ball, $v = Ft/m$.

On the other hand, the angular velocity of ball's rotation will decrease under the action of the torque generated by the same frictional force. The torque is negative (slows down the ball's rotation) and is equal to $-FR$; the resulting angular acceleration is therefore $-FR/I$. After time t, the angular velocity of the ball will be

$$\varpi(t) = \varpi - FRt/I = \varpi - vMR/I$$

Here we used the fact that $v = Ft/m$. We can now recall that at that moment the sliding stops and the ball starts to roll, i.e. $\varpi(t) = v/R$. We conclude that

$$\varpi - vMR/I = v/R \, .$$

Now one can find the final speed:

$$v = \frac{\varpi R}{MR^2/I+1} = \frac{\varpi R}{5/2+1} = \frac{2}{7}\varpi R.$$

CHEMISTRY

5 points:

In 2214, the spaceship *Yury Gagarin* arrived to the planet Didymos that is orbiting the orange dwarf in the Capricorn constellation. This star closely resembles our Sun, but it is too faint to be visible from the Earth, so it has no name yet. In XXII century, using a super-powerful space telescope *Arthur Eddington,* future astronomers will discover eight planet orbiting this star, and the third planet will be dubbed Didymos because of its close similarity to our Earth in terms of composition (iron core, silicate crust, water oceans, oxygen-nitrogen atmosphere), distance from its sun, average temperature, axial tilt, etc. Upon arrival, *Yury Gagarin* crew members made a detailed description of Didymos, which confirmed Didymos was almost an exact copy of our Earth: there were mountains, deserts, atmosphere, clouds there, its oceans and plains were populated by numerous living organisms, etc. However, despite of its name (“didymos” means “a twin”) Didymos appeared to be not a twin of our Earth: Chemical analysis revealed a complete absence of calcium in Didymian crust and oceans.
Please try to predict as many differences between the Earth and Didymos (their lithosphere, hydrosphere, biosphere, etc) caused by the absence of calcium.

Solution

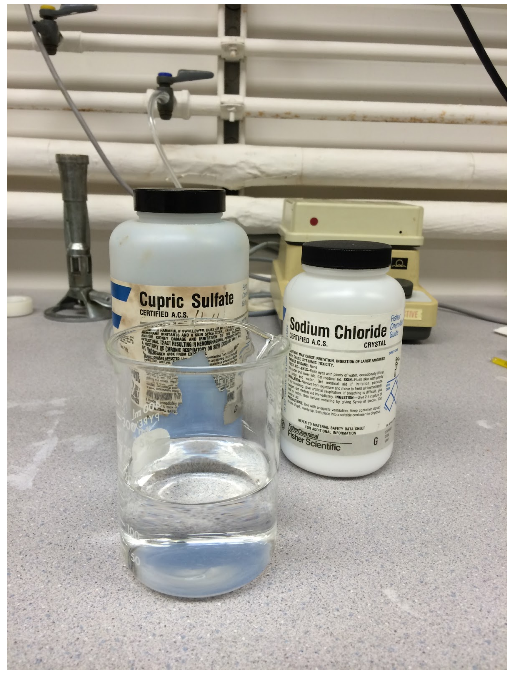
Calcium is one of essential elements, mostly due to its ability to form insoluble salts with silicic, carbonic, and phosphoric acids. Its absence in the Didymian crust would have several important consequences.

1. Volcanic rocks and tectonic processes. The crust and mantle of Earth like planets is composed mostly of silicates (silicic acid salts) of several divalent metals, including calcium. The lack of calcium would mean many abundant minerals, such as anorthite (one of feldspars), would be absent in the Didymian crust. That means there would be no Earth type granites and basalts on the Didymos, so most Didymian rocks would be different from Earth rocks: they would have different hardness, resistance to erosion, melting temperature. That would affect the intensity and the type of volcanism, earthquakes (*didymoquakes*) and tectonic processes on Didymos. In other words, volcano eruptions on Didymos are very different from what we observe on our planet, the shape and the hight of Didymian mountains is also different. In the absence of calcium, the Earth type basalts cannot be formed, which means the Didymian oceanic crust has different density, so the average depth of Didymian oceans is different from the depth of the Earth ocean.

2. Sedimentary rocks. Since calcium carbonates, sulfates, and phosphates are poorly soluble, calcium is one of the most essential components of sedimentary rocks. There will be no limestone, marble (both are calcium carbonate), gypsum (calcium sulfate), apatite (calcium phosphate) on Didymos, which means the mountains like Appalachian mountains would be impossible on Didymos. The absence of limestone and gypsum, which are very abundant on our Earth, would make karst processes impossible on Didymos, so there is almost no caves on Didymos.
3. Ocean salinity. A major part of sulfates on Earth is bound in a form of gypsum (insoluble calcium sulfate). In the absence of calcium, there will be much more sulfates in ocean, that means Didymian marine water contains much more sodium sulfate than sodium chloride. The amount of phosphate in water is also much higher, which has some important consequences for Didymian life.
4. Atmosphere. The calcium carbonate deposits in the Earth crust are huge. Had all carbon dioxide accumulated in the carbonate sediments been liberated to the Earth atmosphere, the atmospheric pressure would increase more than tenfold, and carbon dioxide would become the major component of Earth atmosphere. On Didymos, carbon dioxide cannot be bound by calcium, which means Didymian atmosphere contains much more CO_2 than the Earth atmosphere. Accordingly, due to the greenhouse effect, the average temperature on Didymos is higher that on Earth.
5. Life. The most important consequences for the life is that in the absence of calcium there will be no organisms with carbonate skeleton (most vertebrates), no mollusks (except cephalopods: squids, octopuses, etc), no corals. However, the creatures with cartilaginous skeleton (similar to our sharks or sturgeons) will be thriving due to the absence of competition with bone organisms. Interestingly, since phosphate concentration in the Didymian ocean is much higher, Didymian living organisms have much less problems with minerals to support DNA and RNA synthesis (phosphorus is an essential component of DNA and RNA).

This list of possible consequences of the lack of calcium is not exhaustive; I listed only the most important ones.

10 points:

I did the following experiment (actually, the experiments were done by my assistant, Lena, while I was taking pictures). I took sodium chloride and copper (II) sulfate (10 g of each) and dissolved them in 200 mL of water:	
Then I poured the solution obtained into two conical flasks:	
I took two identical balloons; In one balloon I added 10 grams of zinc powder, in another balloon I added 10 grams of magnesium shavings:	

Then I attached each balloon to the flasks:	
and put the flasks on the scales:	

Zinc powder is in the pink balloon, magnesium shavings are in the blue balloon, and, as you can see, the metals in both balloons are not in contact with the liquid.
The balance is at equilibrium.

Please, describe what will happen when I slightly pull both balloons up, thereby allowing zinc and magnesium powder to spill into the flasks, and then leave both flasks to stay on the scales. The balloons remain tightly connected to the necks of each flasks. Draw equations of the chemical reactions in the flasks.

Solution

At the first glance, the answer is trivial: when copper salt solution is in contact with zinc, magnesium, iron, or other active metal, the exchange reaction takes place:

$CuSO_4 + Zn \rightarrow ZnSO_4 + Cu$

(The reaction with magnesium will be similar). However, we have to take into account the following consideration. Magnesium, as well as aluminium and some other metals, is *very* active. It is so active that it reacts with oxygen at room temperature:

$Mg + O_2 \rightarrow MgO$

We do not see that reaction, because magnesium oxide (MgO) is a very hard and stable solid, which forms a very thin and dense film on the metal surface, thereby protecting the bulk metal from further oxidation. Alumimium behaves similarly, which explains high chemical stability of these two metals. When you try to remove the oxide film from the metal (for example, by filing it), the naked metal comes to a contact with air, and the oxide film immediately form again.

However, when you add a solution of copper salt to the piece of magnesium, copper ions penetrate through small holes in the oxide film and the exchange reaction starts. During this reaction, magnesium atoms go to a solution, and copper deposits on the metal surface, so the oxide film peels off the metal. As a result, naked magnesium metal comes to contact with water. Since magnesium is very active, it reacts with water to produce magnesium oxide and hydrogen. The whole set of reactions in these two flasks is as follows.

1. In the zinc containing flask:

   $CuSO_4 + Zn \rightarrow ZnSO_4 + Cu$

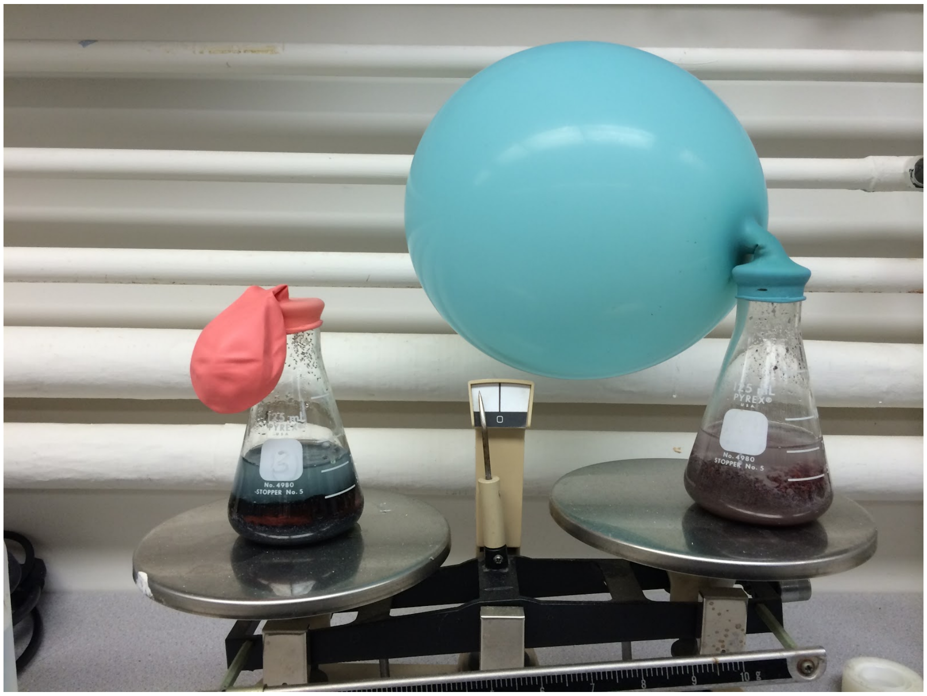
2. In the magnesium containing flask:

   $CuSO_4 + Mg \rightarrow MgSO_4 + Cu$, and

   $Mg + H_2O \rightarrow Mg(OH)_2 + H_2$

(Note, the second reaction goes only in parallel with the first one, because otherwise the surface of magnesium is protected by the oxide film).

The balloon attached to the magnesium containing flask will start to inflate, and this scale will go up (see the photo).



When I was a school student, I used this reaction to obtain hydrogen. I could not use a standard method of obtaining hydrogen, because it was impossible to buy hydrochloric or sulfuric acid in an ordinary shop. Fortunately, copper sulfate was commercially available (it was a component of some pesticides, and it was used for treatment of walls during home renovation). Aluminium wire was easily available too, so, using these materials, I was able to produce almost unlimited amount of hydrogen for my home experiments.
With regard to the role of sodium chloride, it is not completely clear. Probably, it serves as a catalyst.

BIOLOGY

5 points:

Botanists have long been intrigued by plant movements. For example, Charles Darwin, the co-discoverer of the theory of evolution by natural selection, also wrote a book called The Power of Movement in Plants. A few plants are capable of rapid movements, as when a Venus flytrap snaps shut on an unsuspecting insect. Many more plants can move more slowly, visible on the scale of minutes or hours using time-lapse photography. Give three examples of such plant movements and explain their functions.

Answer:

Plants move to respond and adapt to the environment. Their movements generally result from cell elongation and cell division, which can cause different parts of the plant to bend or grow in new directions. Some examples include the following:

1. Phototropism, movement towards light: this is commonly seen in, for example, houseplants growing on a windowsill, whose stems will tilt towards the window in an effort to maximize the light falling on their leaves that drives photosynthesis for the plant.

2. Gravitropism, movement towards gravity: this is seen in plant roots, which grow down to reach water and nutrients; seeds that germinate underground will send their roots downward and their shoots upward, thus exhibiting positive gravitropism in the roots and negative gravitropism (i.e. growth <u>away</u> from gravity and towards the light) in the shoots.

3. Twining, movement around a support. For example, the stems of morning glory plants will rotate around until the tendrils contact a support, after which the tendrils wrap around the support, enabling the plant to grow upwards.

4. Thigmonasty, touch-induced movement: this is the type of movement demonstrated by the Venus flytrap when its leaves snap shut on insects that can be a source of scarce nutrients, or by the sensitive plant (*Mimosa*), whose leaves fold up for protection when they are touched or exposed to temperature extremes.

5. Flower opening: the petals and sepals of flowers open so that flowers can be fertilized (sepals are the leaf-like structures, usually green, around the outside of flowers). Most flowers open once and stay open, but some types of flowers open and close every day. For example, the crocus is one of the first plants to bloom in the spring; its flowers open during warm days but close at night, presumably for protection from the cold.

Time-lapse movies of the above processes can be found at
http://plantsinmotion.bio.indiana.edu/plantmotion/starthere.html

10 points:

In recent years many scientists have studied the microorganisms (bacteria, archaea, and fungi) that inhabit our bodies. All of these microorganisms together are called our “microbiota”, and their collective genetic material is called the “microbiome”. It is known that the total number of cells in the microbiota in one person (about 10^{14}) is about 10 times the number of human cells in the same person. All these microorganisms are likely to be involved in important ways in many of our bodies’ functions. One piece of evidence for the involvement of our intestinal microbiota (also called the “gut flora”) in regulating our bodies’ energy balance is that the microbiota of people of normal weight differs from that of people who are obese.

a. Give at least two possible explanations for this finding.
b. How could you test which explanation might be correct?
c. How could you test the hypothesis that the microbiota of obese individuals could be a contributory factor in the development of obesity in these individuals?

Answer:
A. Give at least two possible explanations for this finding.
1. Obese people eat a different diet from normal weight people, which enables different microbes to grow; or 2. The people who end up as obese may have started with a different microbiota (for some other reason), which extracts less energy from food that the microbes found in normal weight people, causing the obese people to absorb more calories and thereby become obese; or 3. A combination of the first two explanations: Obese people may begin eating a different diet that promotes growth of microbes which use up less of the energy in food and thereby, through a feedback process, contribute to the development of obesity. 4. Finally, it is possible that the influence of the microbiota on weight does not depend directly on the energy absorbed from food by the microbiota, but instead proceeds through other processes. For example, there is some evidence that compounds produced by the microbiota can alter the regulatory system that maintains the body's energy balance. The microbiota may also produce substances that affect the body's inflammatory response, which has been linked to obesity.

B. How could you test which explanation might be correct?
1. Measure the diets and physical activity of normal weight and obese people to see if the calorie intake differs. If it does not, then it could be the microbiota that contribute to the weight difference.
2. Measure the efficiency of their microbes' digesting food. If the microbiota do contribute to the weight difference, one might predict that microbes from obese individuals would break down some otherwise indigestible components of food and make the energy in these components available to their human hosts more than would the microbes from normal weight individuals.
3. Give both normal weight and obese people the same diet and analyze their microbiota over time to see if they become more similar.

C. How could you test the hypothesis that the microbiota of obese individuals could be a contributory factor in the development of obesity in these individuals?
One could try transplanting the microbiota from normal weight to obese individual and vice versa to see if their weight changes. This experiment has already been done in mice and it was found that indeed transplants of the microbiota result in weight changes: when the microbiota of obese mice was put into lean mice, the lean mice

became obese, without a change in their diet. When microbiota were transplanted from lean humans to individuals with metabolic syndrome (a precursor to type II diabetes and often related to obesity), the transplant recipients showed improvements in their responses to insulin.

Another way to alter the microbiota is to treat with antibiotics, which will kill the bacteria. Long-term treatment of humans with antibiotics often causes weight gain. One explanation is that microbes use a significant amount of the energy found in food.

COMPUTER SCIENCE

Solutions must be typed and submitted in one of following formats:
.txt .c .cpp .java .py
Solutions written in Java, C, C++, Python and pseudo-code are accepted.
Pseudo-code guidelines are at
http://users.csc.calpoly.edu/~jdalbey/SWE/pdl_std.html

5 points:

For the New Year's party we need to buy N party hats, but since we're doing the last minute shopping, the 'Party City' has only A gold hats and B silver hats. In how many ways can we buy N hats? (consider all hats of the same color to be the same).

For example, the following input:
6
3
10

should result in the output:
4

Solution:

1) Straightforward:

```
n = int(input())
a = int(input())
b = int(input())
```

```
    ans = 0
        for i in range(0, a + 1):
            for j in range(0, b + 1):
                if i + j == n:
                    ans += 1
    print(ans)
```

2) More mathematical:

```
n = int(input())
a = int(input())
b = int(input())
if a + b < n:
    print(0)
else:
    if a > n:
        a = n
    if b > n:
        b = n
    print(a + b + 1 - n)
```

10 points:

You are given the genomes of two species, and you want to run a quick test to see what is the longest chunk of DNA that they have in common. The genomes are rather large (millions of letters A,C,G and T) and look like the following: "GGACGUUGACGUCGAAACCGGU". Write an algorithm to find the size of the longest common chunk of DNA. More efficient algorithms will get more points [Hint: can you minimize the number of times you scan the genome?]

The output should consist of the length of the longest common region and positions of the first letter of the common region in each of the genomes.

Solution:

```
// DNA segments are entered into arrays S and T
print "Enter the length of the first segment"
input m
S = array(1..m)
print "Enter the length of the second segment"
input n
T = array(1..n)

// L is a two-dimensional array used to hold the lengths of
// found matching regions. Note that each matching pair
// of symbols located at S[i] and T[j] is either a continuation
// of previously found region whole length is stored at
// L[i-1,j-1] or a beginning of a new region
L = array(1..m, 1..n)

// posS and posT will hold the positions of matching regions in
// S and T; len will hold the length of longest common region
posS = -1
posT = -1
len = 0

for i = 1..m
    for j = 1..n
        if S[i] == T[j]
            if i == 1 or j == 1
                L[i,j] = 1
            else
                L[i,j] = L[i-1,j-1] + 1
            if L[i,j] > len
                len = L[i,j]
                posS = i-len+1
                posT = j-len+1
        else
            L[i,j] = 0

 print "Found longest common region of length "+len+" located at
starting positions "+posS+" and "+posT
```