Problem of the Month

POM 2022-2023

November, 2022

MATHEMATICS

5 points: In SigmaLand, there is a very large grocery store that sells apples and oranges. At the start of the day, it stocks 1729 apples and 2022 oranges on its shelves (with a limitless supply in a back room). The SigmaLand customers always come in and blindly grab exactly 2 randomly chosen fruits off of the shelves. (This does not mean they are equally likely to grab apples and oranges. They are more likely to grab the fruit type that there is more of.) The owner, Sigman, restocks the shelves after every customer. Specifically, if a customer buys two apples or two oranges, Sigman adds an apple to the shelves. If the customer buys two different fruits, Sigman adds an orange to the shelves. At the end of the day, when only one single piece of fruit is left, Sigman notes the type of fruit that remains. What is the probability that the remaining fruit is an apple?

Hint: Try working out some smaller examples. What if you start with just a few apples and oranges? Can you spot a pattern? Can you prove it?

Answer: 1, certain, 100%

Solution: The key observation is that the parity of the number of oranges does not change. When removing two identical fruits, the number of oranges decreases by either 0 or 2 depending on the type grabbed, and when removing two different fruits, the number of oranges does not change since an orange gets put back. Since the number of oranges starts even, it must be even at the end, meaning it must be 0, so the last fruit must always be an apple.

10 points: In OmegaLand, there is a street with an address for every real number from 0 to 1. Each address contains an infinite grocery store that sells an infinite amount of apples and oranges. At every store, the portion of all fruits that are apples is precisely equal to the address of the store. For example, store 0 sells only oranges, store 1 sells only apples, and store 0.5 stocks them in equal amounts. Sigman decides to go buy some fruit. He starts by picking a store to go to uniformly at random. Once there, he chooses 2465 fruits to buy by randomly grabbing them off of the shelves, just like the SigmaLand customers. What is the probability that he buys 2022 apples (and 443 oranges)?

Hint: Try working out some smaller examples. What if you are buying just one or a few fruits? If you're having trouble finding the value even for small examples, try reframing the problem in a way that makes it easier. The expressions in this problem can be written in several ways, some of which are almost impossible to evaluate, and others for which are very easy to evaluate.

Answer: 1/2023

Solution: This is a tricky problem. We can solve it with the following trick called "coupling". Let's suppose that each store sorts their fruits on a line, putting the apples first and oranges second. We can call this sorting by "appliness" (in descending order). Stores with large addresses are going to have many apples, so the appliness threshold where the apples turn into oranges are also going to be large. Since he is picking each fruit off of the shelves randomly anyway, Sigman can pick the appliness of the fruits he'll buy ahead of time, and then those fruits will be more likely to be apples at high addresses. For example, if Sigman decides to buy fruits with applinesses of 0.1, 0.3, 0.4, and 0.7, then he'll buy 3 apples if he visits store 0.6, but only 1 apple if he visits store 0.2. This means Sigman is picking 2023 numbers randomly: the 2022 applinesses of the fruits he buys, as well as which store to visit. After sorting the list, the rank of the store's address in that list of numbers will determine how many apples Sigman buys. Specifically, the number of applinesses below the address will equal the number of apples bought. However, the rank of the address in the list is equally likely to be anything from 1 to 2023. Therefore all outcomes from 0 apples to 2022 apples are equally likely, so the answer is 1/2023.

PHYSICS: SPECIAL RELATIVITY

Preamble

Don't be intimidated by the topic this month! Many budding physicists (rightfully so!) have a fascination and reverence for the ideas of special (and general) relativity. The unintuitive nature of the foundational concepts in this field are a gateway to some of the weirdest and most bizarre phenomena in the universe, and bizarre phenomena are precisely our raison d'être as physicists. However, in order to not get confused reasoning about physics we don't have much experience with in everyday life, we need the proper mathematical scaffolding to guide our reasoning. General relativity, unfortunately, requires the full machinery of differential geometry - something a bit difficult to pick up in a matter of weeks. Many ideas in special relativity, however, require nothing more than the language of linear algebra.

The most intense math that this month will require is matrix multiplication - this is a **very** important skill to learn across all STEM disciplines (some of you may remember being emphasized in the Quantum Computation semilab last summer), and all future months will require it in some shape or form. It's not difficult to learn - it's an algorithm that can be memorized by working through a few simple examples in an afternoon.

Special Relativity

I highly recommend watching this short introductory series on special relativity courtesy of Minutephysics, as well as this nice elucidation of matrix multiplication by 3blue1brown.

Some General Background You Don't Actually Need for the Problems:

The development of the theory of electricity and magnetism, culminating in Maxwell's equations for the dynamics of the electric and magnetic field, immediately suggested a surprising result. An oscillating electric field generates an oscillating magnetic field, and an oscillating magnetic field generates an oscillating electric field. Moreover, the speed of the resulting wave can be calculated, and it is always $c = 2.997 \times 10^{8}$ meters per second. Moreover, this value nearly precisely agreed with Rømer's empirical measurement of the speed of light from nearly two centuries earlier.

This marked an incredible breakthrough in our understanding of the world, and immediately offered one of those tantalizing curiosities that often lead to scientific revolutions - a paradox. Maxwell's theory suggested that light *always* moves at speed c, regardless of the observer in question. This find is fundamentally incompatible with Galilean relativity, which would suggest that an observer running at 1m/s in the same direction as the light should observe the beam traveling at $c - 1$m/s instead. This lead to a proliferation of theories about the aether, a notion finally debunked by Michelson and Morley's experiment.

The resolution to this puzzle came in Einstein's 1905 paper - On the Electrodynamics of Moving Bodies. It suggested we must give up the fundamental assumption of how velocities add when observers move relative to each other. To understand what this means, we must introduce the notion of a *coordinate system*. We are used to the coordinate system (x, y, z) in everyday life - the three numbers x, y and z tell us where along the three coordinate axes a particular object resides. Suppose I have two objects, one sitting at (x_1, y_1, z_1) and another at (x_2, y_2, z_2). We calculate distances d between objects via the relationship

$$\Delta d^2 = \Delta x^2 + \Delta y^2 + \Delta z^2,$$

where $\Delta x = x_2 - x_1$, $\Delta x = y_2 - y_1$, and $\Delta x = z_2 - z_1$. The distance Δd between objects should be independent of what coordinate frame we choose, as this is a fake choice of description made by an observer. If my friend had chosen a different set of coordinates in which the objects were instead at positions (x'_1, y'_1, z'_1) and (x'_2, y'_2, z'_2), these two positions *must* satisfy the relationship

$$\Delta x'^2 + \Delta y'^2 + \Delta z'^2 = \Delta d^2 = \Delta x^2 + \Delta y^2 + \Delta z^2.$$

In special relativity, we add in a new coordinate - time. Every vector we consider now has *four* elements, and so are referred to as four-vectors. These vectors are written as (t, x, y, z). The crucial observation is that if the speed of light is constant, in any reference frame, then for any two points along a light beam must always satisfy

$$c^2\Delta t^2 = \Delta x^2 + \Delta y^2 + \Delta z^2.$$

The similarity of this expression to the Pythagorean distance formula suggests a new object that must be independent of coordinate description we choose - the *interval*

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2.$$

It turns out that from this simple assumption, all of special relativity follows! The rule for coordinate transformations must always preserve the interval Δs^2, including the formula for translating between the coordinate descriptions, must *always* preserve the interval.

$$\Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2 \Delta t'^2 = \Delta d^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2$$

More specifically, if we define a new coordinate description (t', x', y', z') from the old one (t, x, y, z) via the relationship

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ \Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 \\ \Lambda^3_0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

This turns out to be quite constraining on the matrix elements $\Lambda^\mu_{\ \nu}$. In particular, our usual low-speed intuition suggest that for an observer moving with velocity v in the positive x direction, we should be able to consider the transformation

$$\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned}$$

However, it is easy to check that this transformation does not preserve the interval $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2$. The correct form of the transformation turns out to be

$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

Where γ is an incredibly useful symbol defined as

$$\gamma := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Note that γ is always strictly greater than 1, and goes to ∞ as v approaches c. Note that we encounter problems if $v \geq c$.

Some Facts that are Crucial to the Problems:

There are a few facts that immediately follow form Lorentz transformations

Length Contraction: An object that is of length L in its rest frame moving at speed v (in the same direction as its length) will have its length contracted to $L' = L/\gamma$. Objects moving at speeds near the speed of light will appear compressed in the direction of motion.

Time Dilation: Objects that move at rapid speeds slow down. If it takes me time T to solve a Rubik's cube at rest, then if I'm in motion then a stationary observer will observe me taking time $T' = \gamma T$ time. Said differently, if observer A is moving at near the speed of light with respect to observer B, A's clock will tick much slower from B's perspective.

Simultaneity: Let's apply a Lorentz transformation to two events that both occur at $t = 0$. The first event takes place at x_1, and the second at x_2. Naively, they both occur at the same time. After a Lorentz transformation, we find

$$t'_1 = \gamma(0 - \frac{vx_1}{c^2}) = -\gamma\frac{vx_1}{c^2}$$
$$t'_2 = \gamma(0 - \frac{vx_2}{c^2}) = -\gamma\frac{vx_2}{c^2}$$

We can immediately see that $t'_1 = t'_2$ *only* if $x_1 = x_2$. If two events occur at the same time in one reference frame, they might not occur at the same time in a different frame!

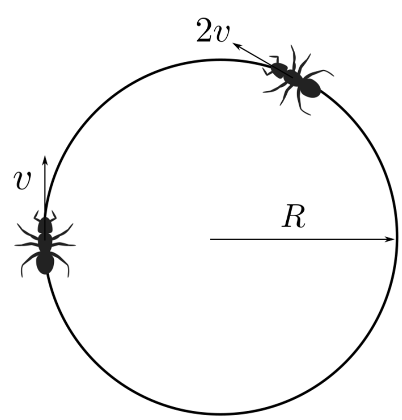
Addition of Velocities: while this fact can be worked out from Lorentz transformations, it is useful to know that if a train is moving at velocity v_1 in the x direction, and a toy car is moving at velocity v_2 in the x direction *in the rest frame of the train*, then the total velocity of the toy car would not be $v' = v_1 + v_2$ as predicted by Newtonian kinematics, but instead will be

$$v' = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}$$

Slower than naively expected!

5 points:

Two ants race around a circular track in opposite directions, as pictured below. These ants are monstrous, and move at significant percentages of the speed of light. Ant 1 moves at speed v, and Ant 2 moves at speed $2v$. Each ant carries a clock which marks off the ant's age. Ant 1's age according to its clock is t_1, and Ant 2's age is t_2. By how much does the difference $t_1 - t_2$ change every time the ants see each other?



Hint: Argue that time dilation is applicable in this setup.

Answer:

Solution: The key observation is that for an *inertial* observer sitting in the center of the ring, ant 1 **always** travels at a speed of v and ant 2 **always** travels at a speed of $2v$. We can then define the Lorentz factors

$$\gamma_1 = \frac{1}{\sqrt{1 - v^2/c^2}}, \qquad \gamma_2 = \frac{1}{\sqrt{1 - 4v^2/c^2}}$$

In this reference frame, call the time interval between ant meetings ΔT. In this time, and 1's proper time changes by $\gamma_1^{-1}\Delta T$ and ant 2's by $\gamma_2^{-1}\Delta T$. The difference in their changes in proper time (equivalent to how much the relative readings of their clocks changes by) is therefore given by

$$\Delta t = \Delta T(\gamma_1^{-1} - \gamma_2^{-1})$$

All that's left is to determine ΔT from the perspective of an inertial observer at rest with respect to the ring. This can be done by noting that the length of the ring is $2\pi R$, and the total speed of the ants toward each other is $3v$. We can therefore write for our final answer

$$\Delta t = \frac{2\pi R}{3v}\left(\sqrt{1 - v^2/c^2} - \sqrt{1 - 4v^2/c^2}\right)$$

10 points:

It turns out that it's not just the space and time coordinates that transform nicely under Lorentz transformations. Energy and momentum also do! In particular, suppose an object is moving at momentum p_x in the x direction, and has total energy E. The transformation properties of these physical quantities for an observer moving at speed v will be

$$p'_x = \gamma(p_x - \frac{vE}{c^2})$$
$$E' = \gamma(E - vp_x)$$

These look quite similar to the transformation properties of x and t! This suggests that just like we arrange space and time into the four-vector (t, x, y, z) we should arrange energy and momentum into the four-vector (E, p_x, p_y, p_z). Just like the interval Δs is conserved, there will be a new invariant quantity we can build from the energy-momentum four-vector:

$$E^2 - c^2(p_x^2 + p_y^2 + p_z^2) = m_0^2c^4$$

The quantity m_0 is the **rest mass** of the object. If the object is at rest, we have the equality

$$E^2 = m_0^2 c^4 \rightarrow E = m_0 c^2$$

Einstein's most famous equation! Just like time dilation, we can read off a useful property of special relativity that an object of rest mass m_0 moving at speed v will have a new total energy

$$E' = \gamma m_0 c^2$$

The Problem:

The Large Hadron Collider fires beams of particles at each other so that they may collide at relativistic speeds. Suppose the particles are all identical and each have rest mass m_0.

Let us restrict our attention to two particles colliding. One particle is moving with speed v in the positive x direction, and the other particle is moving with speed v in the negative x direction.

Part (a) - what is the momentum of *each* particle?

Part (b) - what is the total energy of both particles?

Part (c) - what is the total momentum and total energy of particle 2 in the rest frame of particle 1?

Solution:

(a) The mass of a relativistic particle is given by γm_0, so the momentum of each particle will be $\gamma m_0 v$.

(b) The energy can be determined from the expression

$$E^2 = m_0^2 c^4 + m^2 p^2 \rightarrow E = \gamma m_0 c^2$$

Which we may recognize as a generalization of the famous expression $E = mc^2$. The total energy will therefore be $E_{tot} = 2\gamma m_0 c^2$

(c) Here we must calculate the velocity of particle 2 in the rest frame of particle 1, using a Lorentz transformation. We may do this via the velocity addition formula, which tells us that after a boost by speed v will change the particle's velocity to

$$v_{new} = \frac{v + v_{old}}{1 + v v_{old}/c^2}$$

Both particles travel at the same velocity in the initial frame, which allows us to plug in

$$v_{new} = \frac{2v}{1 + v^2/c^2}, \quad \rightarrow \gamma_{new} = \frac{1}{\sqrt{1 - v_{new}^2/c^2}}$$

Allowing us to determine that the total energy in the new frame is $E = \gamma_{new} m_0 c^2$ and the total momentum is $E = \gamma_{new} m_0 v_{new}$.

Challenge Problem: submit here

We didn't end up posting a challenge problem last month, so this month's problem uses both the ideas of special relativity and electromagnetism!

One of the cool features of special relativity is that it can also tell you how electric and magnetic fields transform into each other under changes of reference frame! The key step is to arrange the electric and magnetic fields into the following *matrix*, called the Electromagnetic tensor:

$$F = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

The correct transformation law under a Lorentz transformation is to multiply from *both sides*

$$F' = \Lambda F \Lambda$$

Part 1) Suppose in one reference frame the world is filled with a constant electric field and zero magnetic field. Find the configuration of electric and magnetic fields seen by an observer traveling in the x direction at speed v.

Part 2) Suppose there is a wire at rest, stretching along the z axis. The wire is neutral. Along this wire flows a current I. We know that the magnetic field due to the wire will be

$$\vec{B}(z) = \frac{\mu_0 I}{2\pi r}\hat{z}$$

(a) Write down the electromagnetic tensor explicitly
(b) Find a Lorentz transformation that takes you to a new reference frame with a *nonzero* electric field
(c) Electric fields can only be generated by a nonzero charge. If the wire is neutral, how is it possible that an observer moving with some velocity observes a nonzero electric field?

CHEMISTRY

5 points:

When Alice came to her lab, Bob, her technician, was looking at three small bottles with barely visible labels. In the first bottle, there was some semitransparent, white and waxy material, in the second bottle, there was a dark red sticky powder, and a small amount of black lustrous crystals were on the bottom of the third bottle.

“Hi, Alice, I am going to discard these three chemicals, because they are unlabeled, but I am not sure about a proper procedure. Can you please advise me how exactly I should dispose of them? It seems at least one of them may be poisonous.”

“Yes, Bob, you are right, every chemical without a label or with an unclear label should be disposed of as chemical hazard waste where it will go through analysis. However, it seems I have an idea on what these three chemicals are, so I propose you to do some experiment. Since they have no labels, we cannot use them for anything serious, but we can use them for fun, can't we? Look.” Alice filled three conical flasks with oxygen (there was an oxygen cylinder in her lab) and labeled them as #1, #2, #3. After that, she put waxy material from the first bottle into a special iron spoon, ignited it and quickly put into the conical flask #1. The waxy material started to burn, the fire was bright and intense, and thick white smoke quickly filled the flask. When the reaction had ceased, Alice poured a little bit of water into the flask and shaked it vigorously. In a few minutes the smoke disappeared, and the liquid remained clear and transparent.

“Did you see it? Now let’s do the same with the other two bottles.” To a big Bob’s surprise, when they repeated the same operations with the materials from the second bottle (they used the flask #2 for that), they observed exactly the same phenomena: violent combustion, white smoke etc. The third experiment (with black powder from the third bottle) gave the same result.

“Great. Now, Bob, please, take a pH paper and test water in each flask. I am sure these liquids are pretty acidic”

Bob put a pH strip into each liquid, and the paper became red-orange.

“How did you know that, Alice?”- Bob asked.

“I know Chemistry”, - Alice replied. After that, she took a sheet of paper, wrote something quickly and put it on the laboratory bench face down.

“Bob, please, do one thing. Take calcium hydroxide solution and barium chloride solution from the shelf, and mix a few drops of each of them with liquids from the flasks numbered 1-3. Do that in separate test tubes. After that, record your observations and compare with what I wrote.”

Question: What did Bob observe when he was adding calcium hydroxide or barium chloride solution to the liquids from the flasks ##1-3, and what was written on Alice's paper?

Hint:

The first word written by Alice is “Precipitation …”

Answer:

Bob observed copious precipitation in all three solutions, and that is exactly what Alice wrote.

Solution:

Obviously, since the same phenomena were observed in all three cases, it is natural to assume that the same chemistry is behind this. However, how is this possible if all three materials look completely different? Usually, when some materials have the same chemical composition but different physical properties, we conclude that we are dealing with some form of isomerism. In this particular case, we are dealing with some special form of isomerism, allotropy. This word is used to describe the various states of an element. For example, oxygen can exist in two allotropic forms, ordinary oxygen (O_2) and ozone (O_3), the latter having a very unpleasant odor and is toxic.

Returning to our case, Alice noticed that the white waxy substance looked suspiciously like white phosphorus. Keeping in mind that it is very toxic and dangerous, she decided to see if this was true. Combustion of phosphorus in oxygen produces thick white smoke due to the formation of solid P_2O_5.

$4P + 5O_2 \rightarrow 2P_2O_5$

This oxide is an acidic oxide, which means that it reacts with water to form phosphoric acid, hence the red color of the indicator paper.

$P_2O_5 + 3\ H_2O \rightarrow 2H_3PO_4$

In addition, most phosphates, with the exception of alkali metal phosphates, are insoluble solids, so when a solution of any calcium or barium salt or their hydroxides is added, insoluble calcium or barium phosphate is formed.

$2H_3PO_4\ + 3\ Ba(OH)_2 \rightarrow 2Ba_3(PO_4)_2 + 3H_2O$

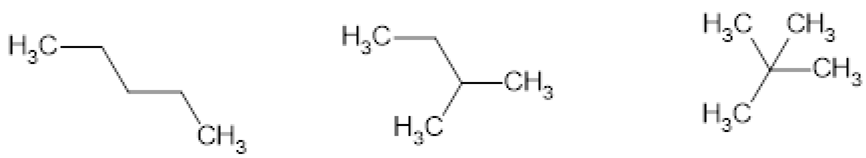
That explains the formation of the precipitate.

We also know that phosphorus can exist in two other allotropic forms, red and black phosphorus. In white phosphorus, the crystals are formed by tetrahedral P_4 molecules, where each atom forms exactly three bonds with three others. Each tetrahedron is bound to other P_4 molecules by weak intermolecular forces, so white phosphorus is soft, fusible, and volatile. And for the same reason, it is very reactive: it oxidizes slowly even at room temperature, and this reaction produces a faint light (hence the name of this element: "phos" means "light" and "fero" means " to bear"). In red and black phosphorus, the P-P bonds form a three-dimensional network, which makes these solids much less reactive and non-toxic. However, since all three solids are made of the same atoms, phosphorus, they produce the same combustion product, P_2O_5, and other two reactions (phosphorus oxide with water, and phosphoric acid with calcium or barium hydroxides) have the same outcome.

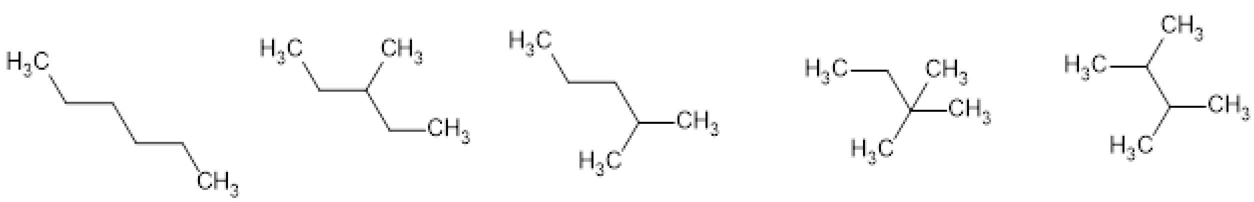
10 points:

We all know that because of carbon's unique ability to form stable carbon-carbon bonds (which are nearly as stable as C-H, C-O, or C-N bonds), including double and triple bonds, and because carbon is tetravalent, isomerism is a very common phenomenon in organic chemistry. Thus, two different compounds with the formula C_4H_{10} can exist, butane and isobutane (a linear and a forked molecules)

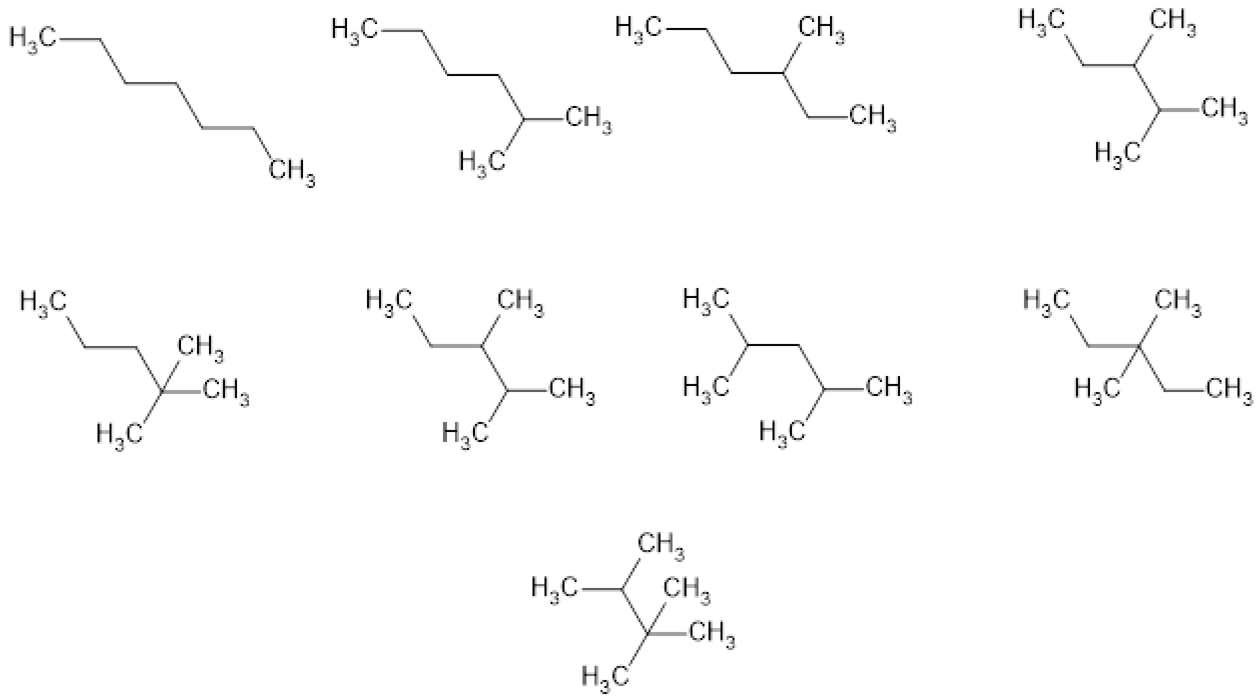
These two compounds belong to the group called "alkanes", and their general formula is $C_nH_{(2n+2)}$. It is easy to see that the number of isomers grows rapidly when *n* increases. Thus, it is 2 for *n*=4, but for *n*=5 it is 3:



For *n*=6, it is 5:

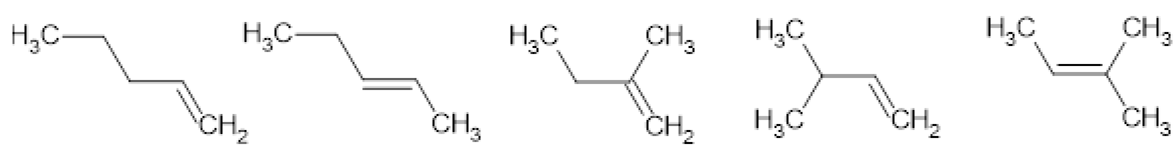


For *n*=7, it is already 9:



And decane (*n*=10) has 75 isomers!

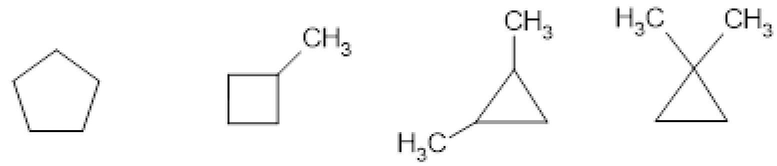
Now consider another class of hydrocarbons with a general formula C_nH_{2n}. One representative is pentene (note “e” in the middle). It has 5 isomers:



1-pentene 2-pentene 2-methyl but-1-ene 3-methyl but-1-ene 2-methyl but-2-ene

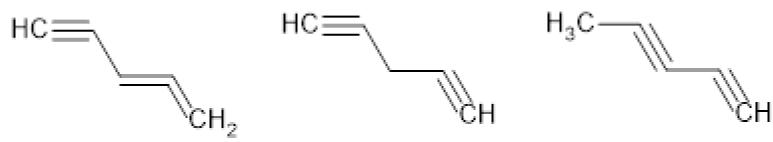
But that is not the end of the story. 2-Pentene can exist in two isomeric forms (E and Z):

and these isomers have different chemical and physical properties. In addition, other hydrocarbons with the same empirical formula (C_5H_{10}) can exist, so called cycloalkanes, cyclopentane, methyl cyclobutene and two isomeric dimethyl cyclolpropanes:



As we can see, there are *at least* 10 different hydrocarbons with a formula C_5H_{10}, whereas pentane (C_5H_{12}) has just three isomers, so removal of just two hydrogens dramatically increases diversity. It is natural to propose that if we take away two more hydrogens (i.e. we are talking about hydrocarbons with a formula $C_nH_{(2n-2)}$) diversity will increase further.

However, if we take too many hydrogens away, the number of possible isomers may decrease. Thus, only three isomers are conceivable for the hydrocarbon with a formula C_5H_4:



(It is necessary to keep in mind that small cycles (8<), with a triple bond cannot exist, and the cycles smaller than 7 carbons with two adjacent double bonds do not exist either, so no cyclic compounds can be proposed for C_5H_4.)

Question:

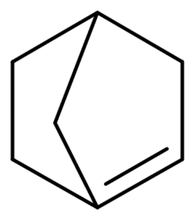
Keeping in mind that cycle formation, and branching are not the only kinds of isomerism, estimate the value of *m* that provides a maximal number of possible isomers in hydrocarbons with a general formula $C_nH_{(2n+2-m)}$. In your analysis, limit yourself with hydrocarbons having 6, 7, and possibly 8 carbons (*n*=6, 7, or 8).

Hint:

Don’t forget about optical and diastereomers.

Answer:

This problem seems to be deceptively simple, and one can expect that some formal solution can be proposed. However, our Math PoM department concluded that modern Math has no adequate tools to solve this problem. The problem is exacerbated with the fact that some structures, which look quite reasonable from the point of view of the valence theory, cannot be prepared in reality. Thus, although this hydrocarbon



looks absolutely reasonable, it is prohibited by Bredt's rule (you may google this term and read more about it), and, therefore, it does not exist.

Therefore, to find the solution, we need, using our knowledge of Chemistry, to enumerate all chemically reasonable structures for each empirical formula. Two approaches may be used. The first approach is the "old school" approach, when you draw each structure from scratch and check (manually) if you foreign anything. The second approach is to look for this information on the Internet. The former is very laborious and requires significant expertise, the latter is easy and simple.

One of the most comprehensive databases of chemical substances is ChemSpider. We can easily search for all compounds with some specific formula, e.g. C_7H_{12}, and after the number of hydrocarbons with each formula is obtained, we just compare these numbers and find the formula with the biggest number of isomers.

It is easy to see that the number of isomers grows when *m* grows (we discuss only even *m*, for when *m* is odd, the hydrocarbon is not a stable molecule, but a free radical or an ion, so we leave odd *m* beyond the scope). The number of isomers of C_6 and C_7 hydrocarbons reaches a maximum when $m = 4$, and then starts to decrease. For C_8 hydrocarbons, the maximal number of isomers is achieved at $m = 6$. This observation can be summarized in some empirical rule: the number of isomers of a hydrocarbon CxHy is maximal when the number of double bonds (and/or cycles) is in between 30-40% of a maximal theoretical value.

It would be interesting to see if this rule is applicable to longer hydrocarbons. You may try to do that by yourself.

BIOLOGY

5 points:

A group of researchers was monitoring the ecosystem of some small river with cold still water, which was populated predominantly by four different species of small fish. The first species (species A) was characterized by a pretty uniform phenotype, which seemed to be quite capable of withstanding potential environmental changes to a degree, and, due to their optimal size and other body parameters, this species was dominating in the ecosystem. The second species (species B) demonstrated a much higher degree of phenotypic heterogeneity, so there were too few survivors in each generation to make species B dominant in the ecosystem.
The third species (species C) was remarkable due to its plasticity over time: the researchers observed significant phenotypic changes in species C in a response to relatively minor variations of environmental conditions. The species D was remarkable because although the species was sedentary, a significant number of individuals demonstrated unusual migratory behavior.

Due to the construction of a dam, the habitat has changed drastically. As a result, one species went extinct, one species survived in the changed habitat, and the other two found new habitats: one species moved to a basin of another small river, whereas another species migrated to semi-saline waters. Name these species, and explain which mechanisms lead to these changes.

Hint:
If you are too successful in some environment, you see no reason to try something new, so you may be totally unprepared for drastic changes in your life.
Answer:

A – Go Extinct
B – Survive
C – Migrate to another small river
D – Migrate to semi-saline water

Species A has chosen a conservative preservation strategy producing off-springs with a set of most optimal traits, which in combination with the large population number should've been a good bet should environmental changes be frequent yet small. Therefore, when a drastic change arrived, no species A fish survived.

Species B has decided not to put all its eggs in one basket come the apocalypse and be ready to have at least a couple of surviving specimens in any possible environmental setting. Although this put them behind species A in the initial conditions, when they changed, their strategy of diversifying their portfolio played well, and species B preserved itself through a few surviving fish.

Species C phenotypic plasticity was most likely supported by their developed sensing mechanism, which allowed them to monitor the environmental cues, such as water temperature, composition, presence of predators, etc. Thus, when the environment had changed significantly and not in a minor way, not allowing species C enough time to adapt, C could sense which direction to migrate to a habitat similar in conditions to their initial one to preserve itself.

Species D migratory nature allowed them to scout the waters and narrow down new possible habitats, not necessarily identical to where most of them initially lived. This bet has allowed them to survive a drastic change by establishing a new colony in a previously identified spot.

10 points:

Anar was traveling for the holidays, and as a result, he caught a cold. He performed a couple of tests and figured out that the cold is of a bacterial nature. To heal himself, he took anti-bacterial antibiotics in the morning and started feeling much better by the evening of the next day. Encouraged, he came to the lab and had a long discussion with his supervisor in his office. The morning right after that, Anar felt sick again, and he immediately informed his supervisor about that.

The supervisor also got sick the next day. He took the same anti-bacterial medicine, and it was efficient … at the beginning. However, the next day, the same symptoms came back, similar to what happened with Anar.

Intrigued, Anar decided to do genomic analysis of his nasal microflora, and he found that both his samples and samples taken from his supervisor show genetic heterogeneity: a fraction of the bacterial population has genes that were sensitive to antibiotics, but some fraction showed resistivity.

Anar was puzzled: whereas bacteria are known to be capable of developing resistivity towards antibiotics, why antibiotic treatment worked well for Anar's supervisor? He asked his friend, who was a microbiologist, but it seems he was busy, so he just mumbled something like "dunno, maybe, bet-hedging?", and left.

What did Anar's friend mean?

Hint:

Just google a couple of articles about bet-hedging and show how they can be used the Anar's story. It will be an interesting and non-trivial reading.

Answer: (solution by Sofia Bachurina)

Anar's friend was referring to a risk-spreading strategy used by organisms in a changing environment. Bet-hedging is when a population develops randomly diverse phenotypes that help it survive major environmental changes but gives up some of its fitness and reproductive success to achieve this (Morawska et al., 2021).

In bacteria, antibiotic persistence is an example of bet-hedging. Bacterial persistence is an epigenetic trait that allows bacteria to survive when encountering antibiotics, but it also decreases their reproductivity under these conditions. Thus, antibiotic persistence can allow some members of a bacteria population (those who are able to switch on the persistence trait) to escape when faced with antibiotics, and these members can keep the species alive. However, because these members cannot reproduce, the bacteria population will not be able to survive in an antibiotic environment for a long period of time.

This is what may have happened to the bacteria that infected Anar's supervisor. When he took the anti-bacterial medicine, the bacteria that did not have persister cells died while those who did survived. However, those who survived could not reproduce before they died, for the medicine provided a prolonged anti-bacterial environment. Thus, there were soon no bacteria left to infect Anar's supervisor. The antibiotic treatment worked well for Anar's supervisor because his bacteria gave up their reproductive fitness in order to survive a bit longer, which is a bet-hedging strategy.

Bacteria can also develop resistance to antibiotics. This occurs when a genetic mutation that makes bacteria resistant occurs within the population. Bacteria with this trait will be more likely to survive and reproduce. So, by natural selection, most of the population would soon also have the resistance trait. Resistance is not a bet-hedging strategy, for bacteria do not lose their reproductive fitness in order to gain resistance.

This is possibly what happened to Anar. A genetic mutation occurred in the bacteria population infecting him that made the bacteria resistant towards the anti-bacterial medicine that he was taking. At first, only a few individuals had this mutation, so Anar started feeling better. Soon, however, these individuals were able to reproduce, and Anar started feeling sick again.

Thus, the difference between bacterial persistence and resistance is what may have caused different responses to the same anti-bacterial medicine in Anar and his supervisor.

References:

Morawska, L. P., Hernandez, Valdes, J. A., & Kuipers, O. P. (2021). Diversity of bet-hedging strategies in microbial communities—Recent cases and insights. *WIREs Mechanisms of Disease*, *14*(2). https://doi.org/10.1002/wsbm.1544

Kussell, E., Kishony, R., Balaban, N. Q., & Leibler, S. (2005). Bacterial Persistence. *Genetics*, *169*(4), 1807–1814. https://doi.org/10.1534/genetics.104.035352

Vogwill, T., Comfort, A. C., Furió, V., & MacLean, R. C. (2016). Persistence and resistance as complementary bacterial adaptations to antibiotics. *Journal of Evolutionary Biology*, *29*(6), 1223–1233. https://doi.org/10.1111/jeb.12864

Veening, J.-W., Smits, W. K., & Kuipers, O. P. (2008). Bistability, Epigenetics, and Bet-Hedging in Bacteria. *Annual Review of Microbiology*, *62*(1), 193–210. https://doi.org/10.1146/annurev.micro.62.081307.163002

LINGUISTICS

We all know that English is a language, and you probably know that American English is different from Australian English or British English, but did you know that American English itself contains many different dialects? Some are dialects found among cultural groups, and some are dialects found in different geographical regions. Some are even both! These dialects can differ between each other both in vocabulary and in pronunciation (or *phonology*).

The International Phonetic Alphabet (IPA) is a standardized method of representing the different speech sounds (or *phones*) of human language. You may have seen the IPA in use before, in a dictionary or at the beginning of Wikipedia pages! Here is a handy interactive guide to the IPA: https://www.internationalphoneticalphabet.org/ipa-sounds/ipa-chart-with-sounds/

5 points:

Given an audio clip of 5 different words said by the same speaker, use the IPA to transcribe the words (as you hear them - meaning the transcription should reflect the regional dialect of the speaker). Please explain your process, and cite any sources that you use. Phonetic spelling without using the IPA will receive partial credit.

The words said are: “square, porridge, mirror, mouth, goose”
The audio clip is located here:
https://drive.google.com/file/d/1bSn05eJ3MOzk-pB70lT73ZiJFn5-2rpU/view

Hint:

Answer:

Square (skwɛeə) or (skwɛɚ) or (skwɛə)
Porridge (pɐɹədʒ) or (pɐɹdʒ)
Mirror (mɛɹɝ) or (mɛɹ)
Mouth (maʊθ)
Goose (gu:s) or (gus)

10 points:

Given audio clips of 5 people from different U.S. regions reading from the same short story, use the IPA to transcribe the words "mirror” and “animal" from each speaker, and match the speakers to the region that they are from. There is only one speaker per region. Please explain your process, and cite any sources that you use. Phonetic spelling without using the IPA will receive partial credit.

Regions: California, Mid Atlantic, New England, New Orleans, New York City

Speaker 1: https://drive.google.com/file/d/17BJnMX4DKTLJvQbikAH29L8noDpgW1zs/view
Speaker 2: https://drive.google.com/file/d/1-HO2g4-qoSQJZiR4RSwZ3ETUm25-2Xbx/view
Speaker 3: https://drive.google.com/file/d/1gEOiiiiV1xvRe_febaazsld99DTfxR0n/view
Speaker 4: https://drive.google.com/file/d/1f-7GEkzwzX7o1oqnBrzd_4Z5DKunI5ei/view
Speaker 5: https://drive.google.com/file/d/1rb_LoW8H4ZWFq6pG31TdfxjcRKdLaCUt/view

Hint:

Answer:

New York City 1: mirror (mɪɹɝ) animal (ˈænəməɫ)
New Orleans 2: mirror (ˈmɪɹə) animal (ænimәɫ) or (ɛænimәɫ)
Mid Atlantic (Alabama) 3: mirror (miɝɹ) animal (ænəmäɫ)
California 4: mirror (mɪɹər) or (mɪɹɛ) animal (ænəmɫ)
New England (Philadelphia) 5: mirror (miɹä) animal (animəɫ)

Solution:

These phonetic spellings were made by comparing the standard translation of the word (using this website: https://www.internationalphoneticalphabet.org/english-to-ipa-translator/) to the audio clip. Using the interactive IPA chart and an IPA reader (http://ipa-reader.xyz/), we modified the standard translation to the regional accent. This is not the only answer, and you may still receive credit for yours even if it's not exactly the same.

All audio clips were taken from https://www.dialectsarchive.com/, which archives recordings of the English language. Check it out if you are interested!

COMPUTER SCIENCE

Thank you to everyone who attended our lecture, and we're so excited to announce our second lecture: **Robotics and Control Theory** by Anna Rosner.
What exactly is control theory? How do you go from an input to a meaningful output? Come learn about how control theory is used to make a system do what you want. No advanced math or computer science knowledge is required!
The lecture will be at **2 PM EST on Saturday, December 3rd**, and you can join here: https://sigmacamp-org.zoom.us/j/83088797287.
Recordings from the last lecture will be posted soon, and we apologize for the delay! The CS POM team has also put together a list of their favorite CS resources, from beginner to advanced, which you can find here, with some recently added resources!

- Your program should be written in Java or Python-3
- No GUI should be used in your program: eg., easy gui in Python
- All the input and output should be via files named as specified in the problem statement
- Java programs should be submitted in a file with extension .java; Python-3 programs should be submitted in a file with extension .py.
  No .txt, .dat, .pdf, .doc, .docx, etc. Programs submitted in incorrect format will not receive any points!

All elements in chemistry can be represented as symbols consisting of one or two characters, where the first letter is capitalized and the second letter (if it exists) is always lowercase. For instance, oxygen's chemical symbol is O, iron's is Fe, and seaborgium's is Sg. Multiple elements can be combined to make molecules, and their symbols are combined to make a chemical formula that represents the amount of each element in that molecule. For example, the formula for table salt, which contains equal amounts sodium (Na) and chlorine (Cl), is NaCl.

When an element appears more than once in a molecule, it can either be repeated or a number can be appended to the element's chemical symbol to indicate how much of this element is present. For instance, HHO is the same as H2O. Additionally, there might be multiple of one molecule or element, in which case the coefficient is placed at the front of the chemical formula. For example, 2H2O is equivalent to HHO+HHO, and 3O is equivalent to O+O+O.

Molecules and elements can be combined together in chemical reactions to result in a different set of molecules at the end of the reaction, which can be described by a chemical equation. However, the number of elements on both sides of the chemical equation representing the reaction must remain balanced. The same amount of every element must appear in the reactants (the left side of the equation) and products (the right side). To balance an equation, the coefficient at the front of a molecule or element may be changed, but the chemical formula cannot be changed. The following is NOT a balanced chemical equation:
CH4+Cl2=CCl4+HCl

This is not balanced because the number of H and Cl atoms is not equal on both sides (left side has four H and two Cl atoms, the right side has one H and five Cl atoms). The following is a balanced equation:
CH4+4Cl2=CCl4+4HCl
By changing the coefficients of Cl2 on the left side and the coefficient of HCl on the right side to 4, the equation is now balanced.

5 points:

Write a program that receives the reactants and products of an equation and determines whether or not the equation is balanced. Both the reactants and products will only contain elements with one letter symbols, and molecules that contain more than one instance of an element will repeat that element's symbol (e.g. HHO instead of H2O). Your program should receive the input file **input.txt**, which will contain the reactants and products separated by plus signs on two separate lines.
Example input file:

```
CCCCCCHHHHHHHHHHHHOOOOOO+6O
6CO+6HHO
```

Your program will produce the output file **output.txt**, which will contain "VALID" if the equation is balanced, and "INVALID" if the equation is unbalanced.
Example output file:

```
VALID
```

In the example above, both sides of the equation contain 6 carbon (C), 12 hydrogen (H), and 12 oxygen (O), so it is balanced.

Solution:

Python:

```python
with open(f"input.txt") as input_file:
   left = input_file.readline().strip()
   right = input_file.readline().strip()

#checks the total amount of each atom
def total(expression):
   counts = {}
   multiplier = 1
   coefficient = ""
```

```
    for c, n in zip(expression, expression[1:] + " "):
        if c == '+':
            multiplier = 1
        elif c.isdigit():
            coefficient += c
            if not n.isdigit():
                multiplier = int(coefficient)
                coefficient = ""
        elif c.isalpha():
            if c in counts:
                counts[c] += multiplier
            else:
                counts[c] = multiplier
    return counts

#If the atom totals are equivalent on both sides, it is balanced
answer = "VALID" if total(left) == total(right) else "INVALID"

with open("output.txt", "w") as output_file:
    print(answer, file=output_file)
```

10 points:

Write a program that receives a list of reactants and products and determines if the reactants could be combined to produce the products (i.e. can the chemical equation be balanced). Products and reactants may contain two letter elements, and if an element appears more than once in a compound, it will be followed by the number representing its amount. Your program should receive the input file **input.txt**, which will contain the reactants and products on two separate lines, with each molecule or element separated by spaces.

Example input file:

```
C6H12O6 O2
CO2 H2O
```

Your program will produce the output file **output.txt**, which will contain “VALID” if the reactants can produce the products, and “INVALID” if it is impossible.

Example output file:

```
VALID
```

In the example above, the reactants contain 6 carbon (C), 12 hydrogen (H), and 8 oxygen (O). The products contain 1 carbon (C), 2 hydrogen (H), and 3 oxygen (O). The balanced form of this equation is C6H12O6+6O2=6CO2+6H2O.

Solution:

This month we chose to feature an exceptionally well crafted and documented solution by Andrew Muratov.

Python:

```python
# reading both lines of reactants and producers
def read_input(file_name):
   with open(file_name) as file:
       reactants = file.readline().strip().split()
       products = file.readline().strip().split()
   return reactants, products



# a simple function for combining the atom counts of two dictionaries
def combine_counts(segment1, segment2):
   combined_count = {}
   for segment in [segment1, segment2]:
       for element in segment:
           if element in combined_count:
               combined_count[element] += segment[element]
           else:
               combined_count[element] = segment[element]
   return combined_count



# count the molecules present as they are written in the input reactant or product formula
def count_molecules(formula, start):
   current_element = ""
   count_elements = {}
   multiplier = 1

   while start < len(formula):

       letter = formula[start]
```

```
        # check if there's a leading digit to set the multiplier for the current element
        if letter.isnumeric():
            multiplier = int(letter)
            start += 1

            # check if the number is 2 digits or more, update the multiplier
            while start < len(formula) and formula[start].isnumeric():
                multiplier = multiplier * 10 + int(formula[start])
                start += 1

            count_elements[current_element] = multiplier
            current_element = ""

        # uppercase letters should start the count of a new element
        elif letter.upper() == letter:
                # add the previous current element a count of 1 if it didn't have a
multiplier
            if current_element != "" and current_element not in count_elements:
                count_elements[current_element] = multiplier
            elif current_element != "" and current_element in count_elements:
                count_elements[current_element] += multiplier

            multiplier = 1
            current_element = formula[start]
            start += 1

        # lowercase letters should be added to the name of the current element
        elif letter.lower() == letter:
            current_element += formula[start]
            multiplier = 1
            start += 1

        else:
            print("Error 1")
            break

    if current_element != "":
        count_elements[current_element] = multiplier
```

```
    return count_elements


# function for determining how many of each molecule exist on one side of the reaction
def count_multipier_joiner(counts, multipliers):
    combined_count = {}

    if len(counts) != len(multipliers):
        print("Error 2")
        return None

    # for all the parsed molecules loop over and multiply them to the set multipliers we are given
    # e.g 3 * H2O + HO should produce {H: 7, O:4}
    for i in range(len(counts)):
        for element in counts[i]:
            if element in combined_count:
                # add to what we already have
                combined_count[element] += multipliers[i] * counts[i][element]
            else:
                # make a new entry with the count and multiplier product
                combined_count[element] = multipliers[i] * counts[i][element]

    # return the final count of all molecules based on the given multipliers
    return {k: v for k, v in sorted(combined_count.items())}


# go through all combinations of multipliers through poping and appending to a stack
#  we will use all these multipliers as output to later compare solutions until we get one that matches
def multi_dim_search(start, max_depth):
    queue = [(start, 1)]
    visited = set()
    while queue:
        node, depth = queue.pop(0)
        if depth < max_depth:
            for i in range(len(node)):
                neighbor = node.copy()
```

```
                neighbor[i] += 1
                # we don't want to accumulate duplicate multiples configurations
                if tuple(neighbor) not in visited:
                    # with each iteration we increase the depth by 1
                    queue.append((neighbor, depth + 1))
                    visited.add(tuple(neighbor))
        # visiting max depth stops us from adding more neighbors to the stack
    return visited


reactants, products = read_input('input.txt')

#get the atom counts inside the molecules for the reactant line of the file
reactant_counts = [count_molecules(section, 0) for section in reactants]



#get the atom counts inside the molecules for the product line of the file
product_counts = [count_molecules(section, 0) for section in products]

#do multiplier enumeration up to depth 20 to get a large number of multipliers we can
assign to our reactant molecules
reactant_start = [1 for i in range(len(reactant_counts))]
multiplier_options = multi_dim_search(reactant_start, 20)

#apply all the multipliers to the reactant counts and get a final atom count for each
combination of the reactants and multipliers
reactant_possibilities = set(
       [str(count_multipier_joiner(reactant_counts,  multiplier))  for  multiplier  in
multiplier_options])

#print(reactant_possibilities)

#do multiplier enumeration up to depth 20 to get a large number of multipliers we can
assign to our product molecules
product_start = [1 for i in range(len(product_counts))]
multiplier_options = multi_dim_search(product_start, 20)

#apply all the multipliers to the product counts and get a final atom count for each
combination of the products and multipliers
```

```
product_possibilities = set(
        [str(count_multipier_joiner(product_counts,  multiplier))  for  multiplier  in
multiplier_options])

# if there is a union between reactant final counts and products final counts, we have
a formula that works
common_formulas = reactant_possibilities.intersection(product_possibilities)

with open("output.txt", "w") as file:
   # if the size of the union is greater than 0, there exists a solution
   if len(common_formulas) > 0:
       file.write('VALID')
   else:
       file.write('INVALID')
```