SigmaCamp's *Problem of the Month* Contest

DECEMBER 2024

Starting from September 2024, we are requiring all submissions to be `.pdf` files (except for CS, which requires `.py` or `.java` files). If you are using Word, you may export to PDF by clicking File > Export > Create PDF/XPS Document.

Mathematics

For all mathematics problems, please provide full justification. **Do not include any code** in your submission — all code submissions will be awarded no points.

5 points:

You have probably noticed that a square of any number ending in 5 ends in 5 (e.g. $15^2 = 225$) and a square of any number ending in 25 ends in 25 (e.g. $125^2 = 15625$).

Similarly, we can observe that S

- since $6^2 = 36$, a square of any number ending with 6 will end with 6;
- since $76^2 = 5776$, a square of any number ending with 76 will end with 76;
- since $376^2 = 141376$, a square of any number ending with 376 will end with 376.

Explain how to continue the sequence $a_1 = 6, a_2 = 76, a_3 = 376, a_4 = 9376$ indefinitely without trial and error, using at most one multiplication to calculate a_{n+1} from a_n, and find a_{14}. (You can use Wolfram Alpha to multiply long numbers exactly.)

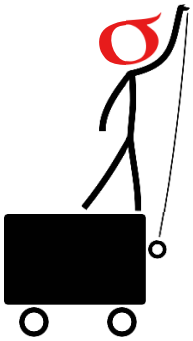
10 points:

A function $f(x_1, \ldots, x_{2023})$ is defined by

$$f(x_1, \ldots, x_{2023}) = \sqrt{x_1^2 + 1^2} + \sqrt{x_2^2 + 2^2} + \sqrt{x_3^2 + 3^2} + \cdots + \sqrt{x_{2023}^2 + 2023^2}.$$

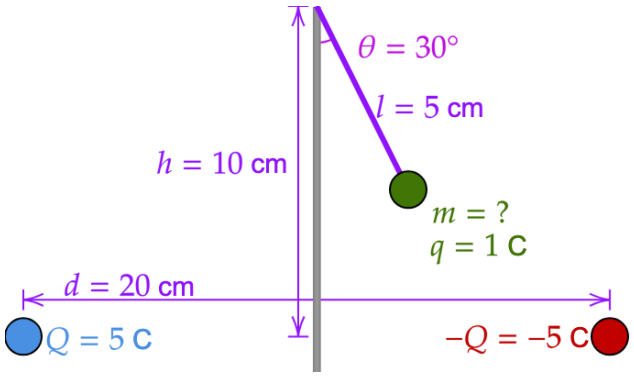
We know that the minimal value of f when $x_1 + x_2 + x_3 + \cdots + x_{2023} = k$ is 2023×2024. Find k.

Physics



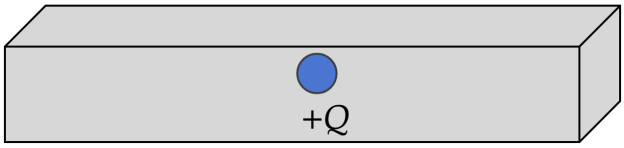
5 points:

Two oppositely charged metal balls are secured 20 cm apart. Exactly halfway between them, a wooden pole is placed. At its top, 10 cm above the centers of the metal balls, a 5 cm-long string is attached, with a metal ball at the end (see diagram). Find the mass of the ball if the angle the string forms with the pole is 30°.



10 points:

A ball with charge $+Q$ is placed at the center of a neutral metal box that is a long rectangular prism.

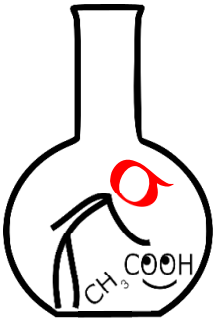


(a) Draw the (approximate) electric field lines and equipotentials in the plane that is parallel to the bottom of the metal box and runs through the charge (don't forget to think about both the inside and the outside of the box).

(b) Now imagine the metal box is grounded. Draw the new electric field lines and equipotentials and find the charge on the metal tetrahedral box.

For each part, please provide a short explanation as to why you drew the field lines and equipotentials the way you did. For example, if the equipotentials were parabolas (which they aren't), you would have to both draw them as parabolas and provide an explanation as to why they are parabolas to get points for that part of the question.

Chemistry



5 points:

Alice, Bob's graduate student, was standing by the bench, gazing at the flasks on the shelf. Bob noticed that Alice appeared disappointed and asked her what was wrong. "Bob, I made a silly mistake. I needed to dissolve the compound I prepared in ethyl acetate, but I accidentally added distilled water instead. Now I have 10 grams of my valuable compound dissolved in 1 liter of water, and I don't know how to retrieve it. I'm worried that I will have to evaporate all the water, which will take more than two days. Do you have any suggestions?"

"Just perform an extraction with ethyl acetate," Bob advised. "Ethyl acetate and water don't mix, so if you add ethyl acetate to your solution and shake it up, your compound will migrate into the ethyl acetate layer, making it easy to separate."

"Yes, I understand that, Bob. However, we only have one liter of ethyl acetate in the lab; I've asked our technician to buy more, but the order will be delivered only after Christmas. I know that the partition coefficient of my compound between water and ethyl acetate is 1, which means by using one liter of ethyl acetate, I can only recover 5 grams of it; the rest will remain in the water. You know that my compound is very expensive, and I cannot afford to lose half of it."

"Not necessarily," Bob replied. "You can extract more. Just do the following: ..."

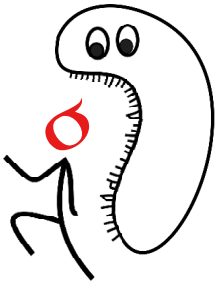
What did Bob say? Explain the procedure and calculate how much of this compound can be recovered.

10 points:

When Alice entered Bob's office, she noticed him staring at the computer screen, looking embarrassed. "Alice, I know you often use ChatGPT for your work. Is it functioning well?" "Yes, Bob, it's a fantastic tool," Alice responded. "Hmm, I asked it to solve a straightforward problem, but something went wrong. Take a look," he said, gesturing for her to see the screen. The problem stated: "An equimolar mixture of two rubidium halides weighing 4.52 g was treated with an excess of silver nitrate. A precipitate weighing 2.87 g was formed. Determine which halides were present in the mixture." ChatGPT's conclusion was: "The mixture likely contained RbBr (rubidium bromide) and XX" "Did you see that, Alice?" Bob asked. "It seems like it can't solve the problem since 'XX' isn't a meaningful answer. Either ChatGPT isn't as smart as you think, or the problem was poorly formulated." "Unfortunately," Alice replied, "ChatGPT missed a detail. The problem does have a solution: it's important to remember that..."

What did Alice say, and what is the answer?

Biology



5 points:

The thermal stability of proteins indicates that many of them have a denaturation temperature just slightly above 42°C. Why is the denaturation temperature so close to our body temperature, and is there any underlying reason for this?

10 points:

Proteins are long polymers made up of amino acid residues, each with distinct side groups. The type of side group determines the possible interactions, leading proteins to fold in ways that maximize these interactions. This results in each protein having a unique shape and function. When synthesized in ribosomes, the nascent polypeptide chain is not properly folded; this occurs later in the process. The chain undergoes thermal fluctuations, with most bonds capable of rotating. Fig. 1 shows the bonds in a short peptide that can easily rotate producing different conformations. More stable rotamers are retained, while less stable ones continue to fluctuate. Ideally, a protein should explore all potential conformations (where each bond can rotate at various angles, typically 0, 120, and 270 degrees). However, the number of possible combinations is enormous: if each amino acid residue has three rotatable angles, a dimer has 9 combinations, a trimer has 27, and so forth. The peptide shown in Fig.1 is a very simple example, but even this short peptide has 9 rotatable bonds, each of which can exist in three different rotational states, so the total number of conformations is 3^9. For a protein made of 100 amino acids, the number of combinations is immense (nearly 3^{300}. Taking into account that at least a nanosecond is needed to probe each conformation, the number of possible conformations exceeds the number of nanoseconds that have elapsed since the Big Bang.

This implies that we face a paradox: if proteins fold as a result of random thermal fluctuations, they ought to do so over a timescale similar to the age of the Universe.

How would you address this paradox?

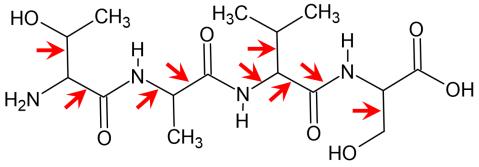


Figure 1: A simple tetrapeptide. Red arrows show rotating bonds.

Linguistics & Applied Sciences

5 points:

In many languages, names of things, such as animals, tend to be based on a combination of more basic words that describe the new concept. These names can describe animals in a way that is useful and easy to use in conversation, even if the name isn't biologically accurate (such as the "seahorse").

Below is a list of 11 sea species and their names in an unknown language:

1. seka mage	A. winged oyster	7. tabui raurau	G. humpback whale
2. seka leuasausau	B. long-snouted coralfish	8. miinana	H. scorpionfish
3. tibi duladula	C. hairy crab	9. diba sonasona	I. partridge tun
4. mosimai	D. horned helmet	10. buburaki seniibi	J. thorny oyster
5. tabui ndisi	E. broad-barred toadfish	11. diba laue	K. spider crab
6. tabui sonasona	F. cameo helmet		

Use the following vocabulary words to match the sea creatures with their names. Provide an explanation for each match:

tibi	the tree *Terminalis catappa*	mage	monkey
sona	to have protrusions	bubu	to sink
dula	to be sewn with needle and thread	mosi	pain
sausau	rough, bony	rau	leaf
nana	poisonous	laue	feather
ndisi	red		

10 points:

(a) The list of English sentences on the right was translated into an unknown Oceanian language and scrambled to produce the list on the right. **Match the English sentences to the Oceanian sentences.** Make sure to **show your work** by explaining the meaning of words and grammatical features you find. Feel free to explain parts of a word in your explanation if you find it necessary. The more you show your work or reasoning, the more partial credit you can get for this problem. Even if you think all your matches are correct — explain how you did it!

1	ee ntook mlaag nok	A. It's not good
2	tulkma vte rong late na	B. I will see them
3	nee sla aat meen nik	C. We won't be able to help you
4	va lokp nok	D. My child already got married
5	o masa oror namee mwermwer	E. The knife is still dry
6	nik mdaw lala namee na	F. I gave it to them
7	na vte tek timi nik	G. I'll be able to curse you both
8	va we timi	H. The fire is alive
9	na swur nir nir mat	I. It's not dry
10	slokp nok timi	J. He saw me at home
11	o mwermwer sngor tavool timi	K. You three won't be able to hear me
12	na swur lala kmoor	L. I won't see you
13	na vte la timi aat meen kmoor	M. She's no longer doing it
14	nee stek timi na	N. The baby didn't sleep well
15	nee mtek na a gvoor	O. It's already wet
16	va wow timi	P. You could have done it for me
17	na stek nir	Q. I am sleeping
18	o aav va is	R. I will curse them so they die
19	nee sdaw nok timi	S. You don't hear well
20	na mla aat meen nir	T. A knife is not a toy for children
21	na tngor	U. It's no longer wet
22	o masa va wow mlitee	V. I won't give it to you both
23	kmaa vte breeng late nik	W. He'll give it to you
24	nik srong tavool timi	X. I didn't see her

(b) Translate the following sentences into the unknown language:

i We didn't see her at home
ii I won't be able to marry you
iii I helped them

(c) Translate the following sentences into English:

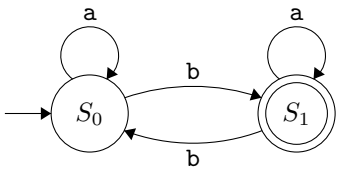
i nee m'mat nok, le nee va is mlitee?
ii nik sdaw tavool timi
iii ee ntook swe timi

Computer Science

This month, you will not be submitting code files (`.py` or `.java`) for CS. Instead, you will be submitting a `.pdf` file for each question.

5 points:

Automata theory is the study of *automata*, which are computational models used in many different areas of computer science. In the September 2023 Linguistics/Applied Sciences POM, we explored how automata theory is used for linguistics (we encourage checking out that problem and its solutions). This month, we will explore an application of automata theory to parsers in programming languages.



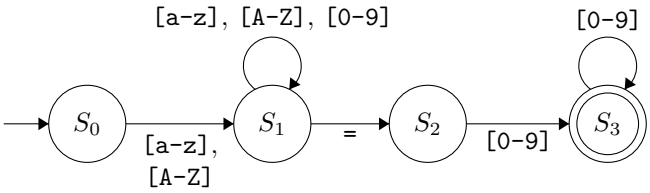
An *automaton* consists of *states* and *transitions* between the states. In the above automaton, the state S_0 is the *initial state*, indicated by the arrow going into it on the left. The state S_1 is a *final state*, indicated by it being double-circled. The transitions are arrows between the states, and are labelled by `a` and `b` in this example. An automaton *reads* a string of characters from left to right and follows the transitions, starting from the initial state. If the automaton finishes reading the string and ends on a final (double-circled) state, we say that the automaton *accepts* that string.

For example, the string "`ab`" is accepted by the above automaton, as the automaton starts at S_0, reads `a` and loops back to S_0, then reads `b` and goes to S_1, which is a final state. Some other strings that are accepted by this automaton are: `bbb`, `baaa`, `bbabbab`. Some strings that are *not* accepted are: `bb`, `bbabab`, `bbababababaab`, and the empty string. After a bit of thought, we can see that the above automaton accepts a binary string if and only if the number of `b`'s in the string is *odd*.

Automata are heavily used in designing parsers for programming languages. Below is an automaton that will accept valid Python declarations of integers of the form

`<variable name>=<digits in 0-9>`

where the variable name can contain letters and numbers but does not start with a digit:



In the above automaton, transitions with the label `[0-9]` allow any digit from 0 to 9. Similarly, `[a-z]` refers to any lowercase letter, and `[A-Z]` refers to any uppercase letter. Some strings accepted by this automaton are (check it for yourself!):

`a=5` `myVariable=0` `num123=0401` `hi12bye34=56565656`

Some strings that are *not* accepted by this automaton are (check it for yourself!):

`1var=3` `num=` `=3` `mystring="hi"` `var=A5` `my_num=3`

Note that the strings `mystring="hi"` and `my_num=3` are not accepted since the quotation mark `"` and the underscore `_` are not valid characters for the above automaton.

For this question, you will **design and draw an automaton** that accepts the first line of Python function declarations, which are of the format

`def <name>(<params>):`

The `<name>` field can consist of letters and numbers, but cannot start with a number. The `<params>` field can contain zero, one, or multiple comma-separated parameters, which each can consist of letters and numbers but cannot start with a number.

For example, some strings your automaton should accept are:

- `def myFunc():`
- `def gcd(a, b):`
- `def func1(myParam5):`
- `def funnyFunction123(num1, num2, num3, num4, num5):`

Some strings that are should *not* be accepted are:

- `def 1func():` *(the function name cannot start with a digit)*
- `func(a, b):` *(the function declaration must start with "`def `")*
- `def thing(` *(the declaration must have parentheses and end with a colon)*
- `def gcd(1a, 1b):` *(parameter names cannot start with digits)*
- `def func(,):` *(the interior of parentheses should be empty if there are no parameters)*

Make sure your automaton includes an initial state and a final state, and test your automaton on the above examples and come up with some other ones to ensure that it is correct. **If desired, your automaton may have more than one final state.**

10 points:

> For the following problem, you should submit a `.pdf` with your answers including any code you write and any plots. **Submissions that only submit code will not receive full points!**

Oh no! A dangerous storm is heading for SigmaCamp! Mark instantly runs to the chemistry lab and starts brewing a potion that will protect the camp from the approaching storm. To make the perfect potion, four ingredients need to be combined in precise amounts.

The potency of the potion is determined by the sum of the strengths of the ingredients used in the potion, and Mark's goal is to create a potion whose potency is as close as possible to the required target value T, using any *integer* combination of the four ingredients. Each of the four ingredients has different strengths S_1, S_2, S_3, and S_4 per milligram of the ingredient used. For example, if 3 mg of ingredients 1 and 3 are used and 6 mg of ingredients 2 and 4 are used, then the potency of the potion is $3S_1 + 6S_2 + 3S_3 + 6S_4$.

Your task is to write a Python or Java program to help Mark determine the closest possible potion potency to T that can be created using a combination of the four available ingredients.

Your code should read the input file `input.txt`, which contains a single line containing space-separated values for T, S_1, S_2, S_3, and S_4. It should output to the file `output.txt`, containing a single line of space-separated non-negative integers a, b, c, d that minimize the quantity $|T - (aS_1 + bS_2 + cS_3 + dS_4)|$. If the answer is not unique, output the *lexicographically smallest* list of integers.

Implement your code in the following two ways:

(a) First, implement your program using the naive approach of iterating through all possibilities of sums. Include the code in your writeup, and give a high-level overview of your code.

What is the relationship between your program's runtime and the value of T? Is the time proportional to T, T^2, or something else? Why?

Try out different values of T and record how much time your program takes. Plot the values using your favourite plotting software, and include the plot in your writeup.

(b) Now, implement your program using *dynamic programming*. Include the code in your writeup, and give a high-level overview of your code.

What is the relationship between your program's runtime and the value of T?

Try out different values of T and record how much time your program takes. Plot the values using your favourite plotting software, and include the plot in your writeup.

Timing your code

In Python, you can time your code by using `time.time()` from the `time` package, as done below:

```
import time

t0 = time.time()
...
# <insert code here>
...
t1 = time.time()

print("My code runs in", t1-t0, "seconds")
```

In Java, you can time your code using the `System.currentTimeMillis()` function.

Examples

Sample Input 1:

```
50 3 6 7 2
```

Sample Output 1:

```
0 0 0 25
```

Sample Explanation 1:

Note that $25 \cdot 2 = 50$. While many other combinations of ingredients are possible (e.g. `3 3 3 1`, the one given above is the lexicographically smallest one.

Sample Input 2:

```
54 20 15 40 90
```

Sample Output 2:

```
0 1 1 0
```

Sample Explanation 2:

Note that $1 \cdot 15 + 1 \cdot 40 = 55$ is the closest achievable value to 54.