SigmaCamp's *Problem of the Month* Contest

November 2024

Starting from September 2024, we are requiring all submissions to be `.pdf` files (except for CS, which requires `.py` or `.java` files). If you are using Word, you may export to PDF by clicking File > Export > Create PDF/XPS Document.

Mathematics

For all mathematics problems, please provide full justification. **Do not include any code** in your submission — all code submissions will be awarded no points.

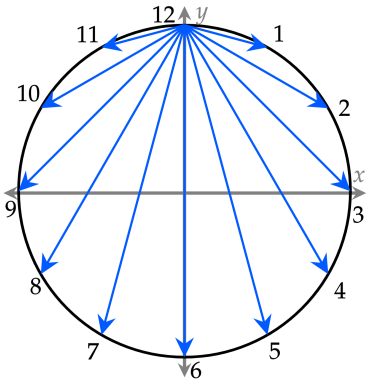
5 points:

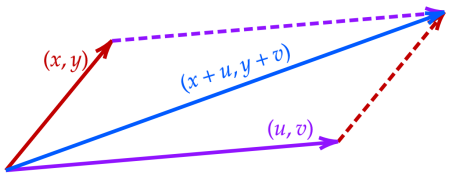
Consider a circular clock of radius 1 foot with 11 vectors* that originate at the 12 o'clock mark and point to the other 11 numbers, as shown in the diagram on the left. Calculate the exact sum of these 11 vectors.

(*) If you have never seen vectors before, this is an opportunity to learn about them. To solve this problem you need to know that vectors in a plane can be represented as arrows. They can be added using the parallelogram rule, as shown in the diagram on the right: place the two vectors at a common starting point and draw a parallelogram where they form adjacent sides — the diagonal from the starting point to the opposite corner is their sum.

Vectors can also be represented in coordinates, like (x, y), where x is the difference in the x-coordinate between the arrow's point and the tail, and y is the difference in y-coordinates. In this representation, vector addition works like this: $(x, y) + (u, v) = (x + u, y + v)$.

To learn more, we recommend Khan Academy (you don't need the whole unit) or an LLM, like Learn About.





Parallelogram rule

10 points:

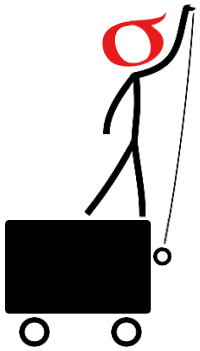
Last Halloween, 5 campers decided to dress up as the original emotions from Inside Out (Joy, Sadness, Fear, Anger, and Disgust). Without coordinating with each other, they each randomly selected an emotion from the film to dress up as.

The campers do not want to go trick-or-treating as a group that contains more than one copy of any costume. They agreed that they would split off into several groups to make sure that all costumes in a group are different, but the campers want to split into as few groups as possible.

(a) What is the probability that the campers were all dressed as different emotions and were able to go trick-or-treating as one group?

(b) What is the probability that the campers had to split up into exactly 2 groups? To help you check your answer, please list all other situations that may occur (up to and including all 5 campers wearing the same costume and splitting into 5 groups), calculate their probabilities, and confirm that all the probabilities add up to 1.

Physics



Five astronauts are on a long-term mission in deep space.

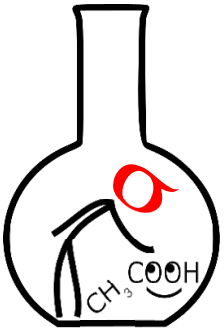
5 points:

In their spaceship, there are two rooms. One of them is near the engine, which generates heat, but otherwise they have the same volume and are connected by an open door. Which room has the greater mass of air? There is no air flow out of or into the rooms.

10 points:

Their rocket breaks down and they must leave the ship to fix it. The spaceship has an air lock. Initially, the air lock, which has volume v, has no air inside it. When the astronauts leave the spaceship, air from the space ship is used to slowly fill the airlock. The space ship is initially at pressure p and has total volume V (including the airlock). Unfortunately due to malfunction, every time the astronauts leave the airlock all of the air inside it is expelled into outer space. What will be the pressure inside the spaceship after each of the 5 astronauts leaves the spaceship? Assume only one astronaut can use the airlock at a time and that the temperature in the spacecraft is kept at constant temperature T (due to a heating system powered by solar panels).

Chemistry



5 points:

There is a hydrocarbon that is a gas under standard conditions. It quickly reacts with bromine (even in the absence of light) and with a $KMnO_4$ solution. This gas was mixed with oxygen in a 1:10 ratio, and the mixture was placed in a steel container and ignited. After the mixture cooled to 0°C, the water formed by the reaction was collected and measured to be 720 mg. The gas remaining in the vessel was pumped through an excess of a $Ca(OH)_2$ solution, and the precipitate formed was collected, dried, and its mass was measured. The mass of $CaCO_3$ formed was 4 g. What is the name and formula of the gas? Explain your answer.

10 points:

Alice found a bottle with some white powder in her lab. The label was partially peeled off, and she was able to read only

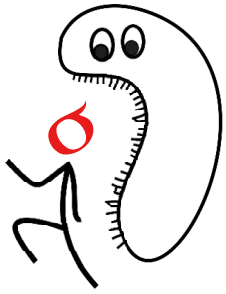
> ...lecular mass 380...
>
> ...rmula $Na_4C_{10}H_{12}$...

She took 4 g of this powder and dissolved it in 100 ml of water to give a colorless basic solution. Then Alice did the following steps, in order:

1. 10 ml of 1 M Na_2SO_4 is mixed with the solution. There is no visible change.
2. 10 ml of 1 M $CaCl_2$ is added to the previous mixture. There is still no visible change.
3. 20 ml of 1 M $CuCl_2$ is added to the previous mixture, forming a precipitate. When it is filtered out, the precipitate is white and the remaining solution is blue.

Give a possible identity for the mystery substance, or explain why the observations above are not possible.

Biology



5 points:

On August 14, 2065, Bob, a brilliant genetic engineer and Alice's colleague, was suffering from anxiety and depression, he was gradually losing interest in his work, and Alice often noticed him staring at a map of Connecticut hanging on the wall opposite his desk. A few days ago, he disappeared. He was not answering his phone. Alice found a mysterious vial on Bob's desk with the inscription "Please read me!!!" and two test tubes labeled "primers". In Bob's lab notebook, Alice found the sequence:

> 5'-GCCGTGCTCATCCTCTTCATGGAACACATCATGATCTCTTCCTCTATCGGTATGGCTTGTGC
> TATGCCCTAGCGCGCCTCGTGCCCATGTTATCTGCATCTGATACTGGCAAGCTTGAGTCTCGTA
> GAGGGGGGTAGAATTCCAGGTGTAGCGGTGAAATGCGTAGAGATCTGGAGGAATACCGGTGGC
> GAAGGCGGCCCCCTGGACGAAGACTGACGCTCAGGTGCGA-3'

and a note:

> Primers:
>
> 5'-GTGCTCATCCTCTTCATGG
>
> 5'-CACGAGGCGCGCTAGG

Alice concluded that this might be a message from Bob and decided to use polymerase chain reaction (PCR) to identify it. She ordered primers with the sequence specified in the note, mixed them with the contents of the mysterious vial, added other auxiliary reagents, and after 50 cycles of the PCR reaction, she obtained a significant amount of product (a fragment of double-stranded DNA).

Alice sequenced this DNA using a nanopore DNA sequencer and obtained a sequence that apparently encoded some kind of short protein (it began with a methionine codon, which is a standard start codon, and ended with a standard stop codon). And after Alice translated the nucleotide sequence into an amino acid sequence (using a standard table of single-letter codons), she smiled and breathed a sigh of relief.

"Oh, I know where Bob is!"

Explain how Alice knew where Bob was.

10 points:

Alice, who was a researcher in Bob's lab, tried to enzymatically synthesize a 51-nucleotide-long fragment by mixing the template strand

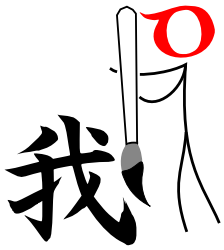
5'-AGGTACACACCGTATATTCAACAAGTACTTCCTTCGACGCGTAGCTACGCG

with a 14-mer primer 5'- CGCGTAGCTACGCG, four deoxynucleotide triphosphates, and DNA polymerase.

To her big surprise, instead of a 51 nucleotide long product, she got mostly a longer DNA (about 88 nucleotide long), along with a small amount of the 51-mer. Alice was very disappointed, but Bob told her that the result was quite predictable and that the problem could be resolved. "By the way, if you repeat this reaction again, but add no XXXX, the major product will be the same", Bob said.

Explain why Alice's experiment failed and what Bob meant by XXXX. What is the sequence of the product obtained by Alice, and how can Alice fix the problem?

Linguistics & Applied Sciences



5 points:

The sample corpus below phonetically transcribes select words from an unknown language: (Note: /x/ is the sound in 'loch', /ɣ/ is /x/ but voiced. /ʃ/, /ʒ/, /θ/, /ʊ/, /ə/, 'y' are English 'sh', 'zh', 'th', 'foot', 'comma', 'yell')

1. wöbʒäyʒä
2. tämä
3. ləlärniyän
4. xawzum
5. ləgzəwba
6. narn
7. trney
8. rmoksa
9. nöbʒzi
10. kuponənawm
11. ʒüdɣü
12. stliɣi
13. tlofk
14. lworngəθämä
15. büsk
16. küpönenawm
17. blemdʒe
18. yiʃün
19. zudɣum
20. wödliɣlö
21. zutx
22. yiʃülelwornk
23. küpönö
24. närniyälle
25. sayəθmak
26. wʊsun
27. lwornk
28. gwomp
29. kupon
30. brmögʒö
31. smusəθrney
32. tlowgum
33. sayəθmagəvüsk
34. lmusəfʃafs
35. nyögʒef
36. ləzdlʊx
37. ksəwba
38. psafs
39. skfombəxupon
40. tmak
41. lmus
42. xəmts
43. dlʊx
44. wöɣemtʃe
45. tmyärnʃä

(a) You notice that certain transcribed sounds appear only in a certain place in the word, such as at the middle or the end of a word or before a vowel. You also notice that certain sounds may frequently be found together in the same word. These sounds can be referred to as "allophones": the actual pronunciation may change, but all of these sounds relate to a single *phoneme*, or distinguishable sound in the language. Make a reasonable estimate of how many phonemes exist in the language presented above and explain why.

(b) Take a look at the language's *phonotactics* — that is, how sounds fit together within a word and in what order. Group the above sounds into categories. How are the phonotactics of this language different from English? Are there sounds that are shared with English but are ordered in a way that isn't found in English?

(c) Write 3 words that obey the phonotactics of the language above and 3 that do not. Explain your reasoning.

10 points:

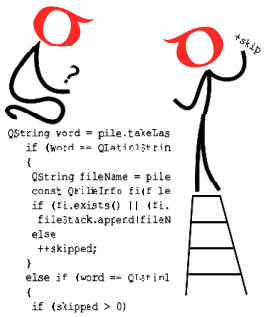
A tiny 4-wheel robot is placed on an airstrip. The airstrip consists of two parallel edges that extend for miles in both directions and is absolutely flat. Both edges are lined with a curb that is taller than the robot itself. The robot can rotate 360 degrees as it moves. It contains touch sensors around the perimeter and a microcontroller that can (1) control the direction and movement of the robot, and (2) store previous position data of the robot. However, it cannot tell its angle with respect to the edge of the airstrip, nor does it know how to turn perpendicularly to the curb. Assume that the robot is placed in a random position on the airstrip and that the width of the airstrip is unknown.

(a) Describe a strategy that will guarantee that the robot hits one of the edges. Try to be as efficient as possible in terms of distance travelled.

(b) If the width of the airstrip is 30 meters, what are the shortest and longest possible distances that a robot could travel using your strategy in part (a) before reaching the curb? What if the width is 300 meters?

(c) Once the robot hits the curb, explain how the robot can keep moving to identify the correct width of the airstrip by hitting the other curb. Try to be as efficient as possible in terms of distance traveled and analyze the best and worst case scenario as in part (b), but in terms of the hypothetical width W.

Computer Science



- Your program should be written in Java or Python-3.
- No GUI should be used in your program (e.g. `easygui` in Python). Other common libraries (e.g. `numpy`, `scipy`) can be used, but **not** in a way that trivializes the problems.
- All the input and output should be done through files named as specified in the problem statement to receive full points.
- Java programs should be submitted in a file with extension `.java`; Python-3 programs should be submitted in a file with extension `.py`. **No `.txt`, `.dat`, `.pdf`, `.doc`, `.docx`, etc. Programs submitted in the incorrect format will not receive any points!**

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5 points:

The Silver Lake Camp and Retreat Center offers a breathtaking night sky perfect for stargazing. One evening, a group of n campers gathered to observe the Milky Way through a telescope. Each camper has identified a star with some brightness recorded as a positive integer (the higher this integer, the brighter the star).

The campers are competitive, and want to find the brightest star they can. Starting with the second camper, each camper i checks with the previous campers $(1, 2, \ldots, i-1)$ to see if any of them observed a star brighter than theirs. The following then occurs:

- If they find a camper who saw a brighter star, then they set their "goal brightness" to the brightness of the brightest star among the first $i-1$ campers, which they will try to attain at tomorrow's stargazing.
- If they check all the previous campers and find no brighter star, then they decide that their star is bright enough, and set their goal brightness to 0.

Write a program that determines the goal brightness for each camper.

Input specifications

Your program should read the input file `input.txt`, which has the following format:

- The first line contains a positive integer n representing the number of campers.
- The second line contains n space-separated positive integers, where the i'th integer represents the brightness of the star observed by camper i.

Output specifications

Your program should produce the file `output.txt`, which contains n integers corresponding to the goal brightness of each camper.

Examples

Sample Input 1:

```
5
8 3 9 10 5
```

Sample Output 1:

```
0 8 0 0 10
```

Sample Explanation 1:

Campers 1, 3, and 4 each have the brightest star when comparing only the campers before them. Camper 2 sees that camper 1's star is brighter, and camper 5 sees that the brightest star among the campers before him has brightness 10, which becomes their goal brightness.

Sample Input 2:

```
9
1 4 100 9 105 39 90 200 54
```

Sample Output 2:

```
0 0 0 100 0 105 105 0 200
```

10 points:

The campers have taken a picture of a small portion of the night sky, which contains n stars. Each star i is positioned at coordinates (x_i, y_i) where x_i and y_i are integers, and has a brightness b_i, which is a positive integer. The campers are trying to find the number of constellations in the picture that are strictly brighter than some threshold.

Constellations are groups of stars that are all connected to each other by a path of "nearby" stars. Two stars i and j, positioned at (x_i, y_i) and (x_j, y_j) respectively, are considered to be "nearby" if the Euclidean distance between their coordinates is less than d:

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} < d.$$

Note that **constellations must consist of at least two stars**. A constellation is brighter than a threshold t if every star in the constellation is **strictly** brighter than t.

The campers are interested in counting the number of constellations of various brightnesses. Write a program that takes in a list of stars and their brightnesses, a distance threshold and a list of brightness thresholds, and computes a list of the numbers of constellations that are brighter than each threshold.

> **! Clarification (November 24, 2024) !** In your program, you should count the number of largest possible constellations that can be formed (e.g. if there is a constellation of 3 stars, that should be counted as a single constellation, rather than four).

Input specifications

Your program should read the input file `input.txt`, which has the following format:

- The first line contains three space-separated positive integers: n, d, m, where:
  - n is the number of stars,
  - d is the distance threshold for stars to be considered "nearby", and
  - m is the number of brightness thresholds.
- The second line contains m space-separated positive integers $t_1, \ldots, t_m$ representing the brightness thresholds.
- The next n lines contain three space-separated integers x_i, y_i, b_i, where:
  - x_i and y_i are the position of star i (**note that x_i and y_i could be negative**), and

– b_i denotes the brightness of star i

Output specifications

Your program should produce the file `output.txt`, consisting of m space-separated integers, where the i'th integer is the number of constellations that is **strictly** brighter than the i'th brightness threshold t_i.

Examples

Sample Input:

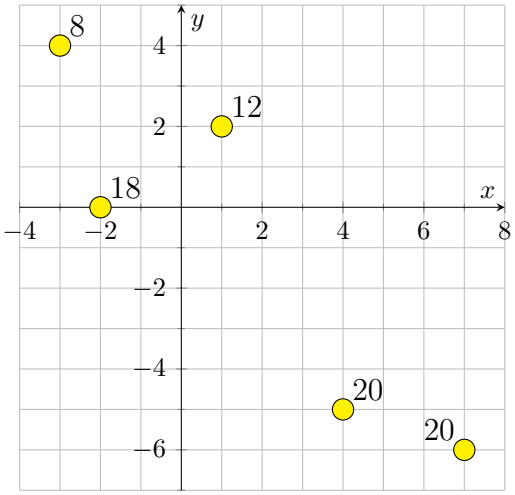
```
5 6 3
5 12 10
1 2 12
-3 4 8
7 -6 20
-2 0 18
4 -5 20
```

Sample Output:

```
2 1 2
```

Sample Explanation:

See a plot of the stars below:



First, disregarding any brightness thresholds, the two stars of brightness 20 cannot be in the same constellation as any of the stars of brightness $8, 12$ or 18, as the star at $(4, -5)$ is more than $d = 6$ away from the stars at $(-2, 0)$ and $(1, 2)$.

For $t_1 = 5$, the stars with brightness $8, 12, 18$ form **one constellation**, and the two stars with brightness 20 form another. Note that we do not count the three pairs of stars (with brightness 8 and 12, 12 and 18, and 8 and 18) as three more constellations. We only count the largest possible constellation of three stars.

For $t_2 = 12$, since the stars at $(1, 2)$ and $(-3, 4)$ do not exceed the threshold (their brightness must be **strictly** greater than 12), they cannot be counted as part of any constellations. Constellations must consist of at least two stars, so the star with brightness 18 is not counted. So, there is just one constellation brighter than $t_2 = 12$, consisting of the two stars of brightness 20.

For $t_3 = 10$, the star at $(-3, 4)$ does not exceed the threshold, but a constellation is still formed by the stars with brightnesses 12 and 18. So, there are two constellations brighter than $t_3 = 10$.