Are All Infinities the Same?

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Are there more

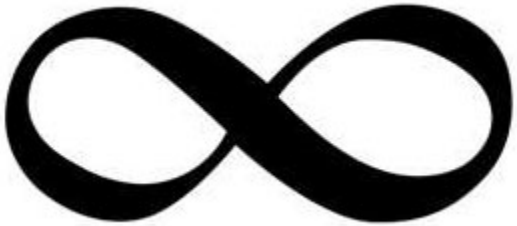
- integers or rational numbers?

1, 25, -170, …

1/3, 371/11,….

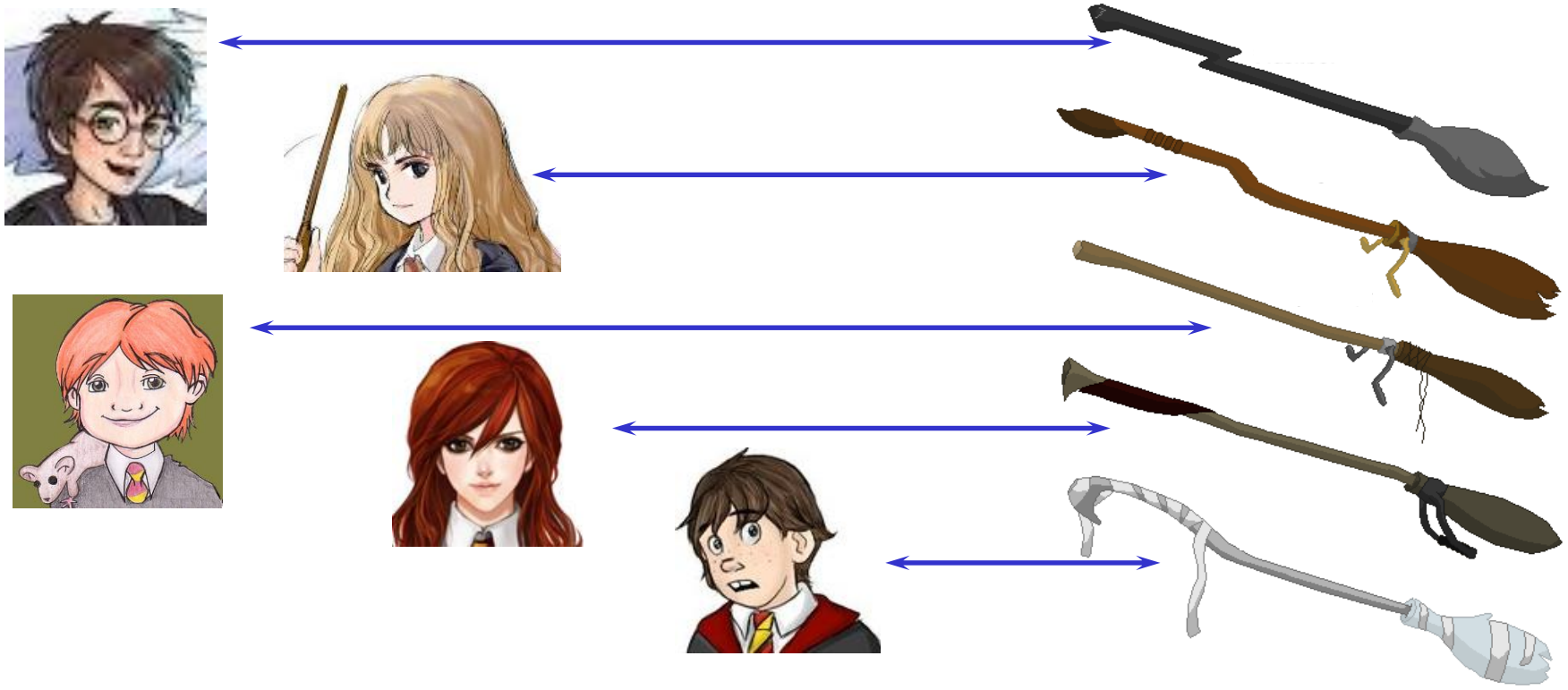
- computer programs or computational problems?

How do we compare infinite sets?



How Do We Compare Sets?

- Question: Are there more students or brooms?

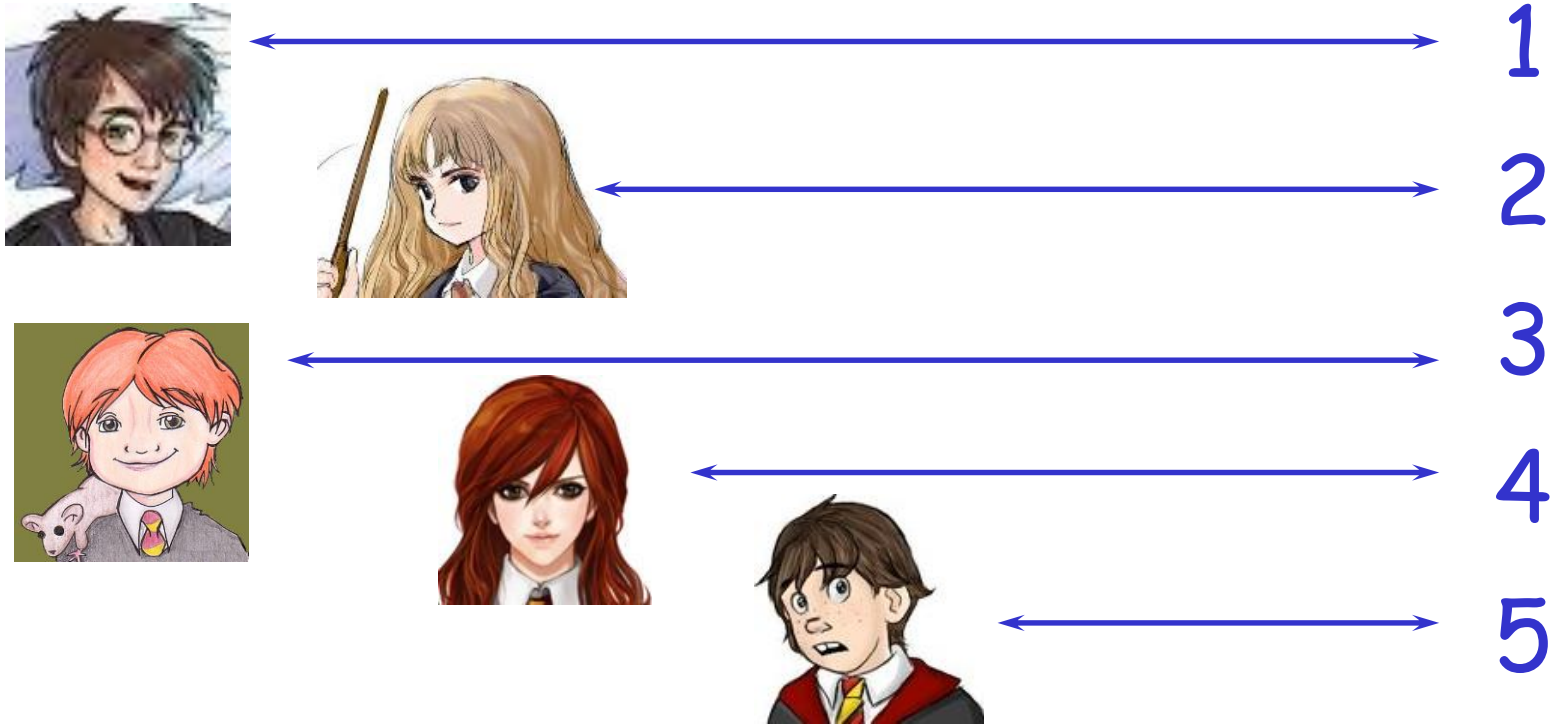


- Answer: The same number.

Find a bijection.

How Do We Compare Sets?

- Question: Are there more students or brooms?



When we count, we find a bijection between items and natural numbers (1, 2, 3, 4, …).

To show that infinite sets have
the same number of items,
we also find
a bijection.

Sets of Numbers

- Natural numbers (numbers used for counting)

  1, 2, 3, 4, 5, …

- Positive even numbers (natural numbers divisible by 2)

  2, 4, 6, 8, …

- Integers

  …, -3, -2, -1, 0, 1, 2, 3, …

- Rational numbers (fractions)

  0, 1, -1, $\frac{1}{2}$, $-\frac{1}{2}$, $\frac{1}{3}$, $-\frac{1}{3}$, $\frac{2}{3}$, $-\frac{2}{3}$, …

- Reals (all numbers on the number line)

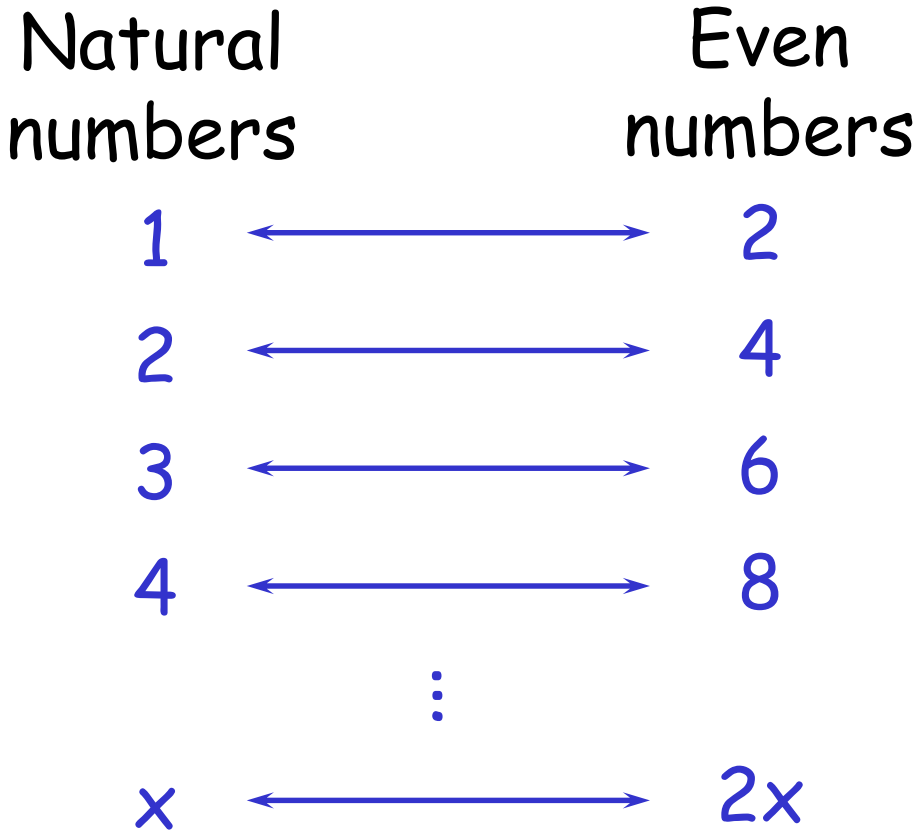
  1.414213562373095…, 3.14159265359…

Are There More Natural or Positive Even Numbers?

More natural numbers:
all positive even numbers are natural, but positive odd numbers are also natural.

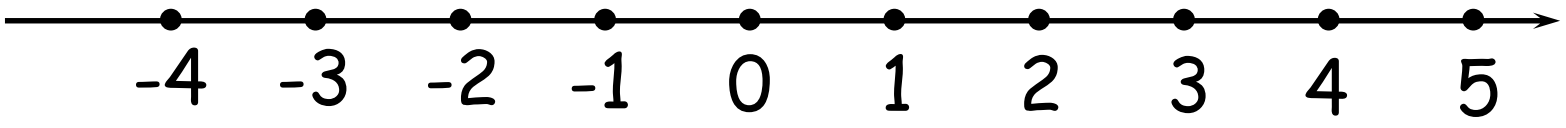
No, the same number.
We just have to find a bijection to match natural numbers and positive even numbers.

Are There More Natural or Even Numbers?



Answer: The same.

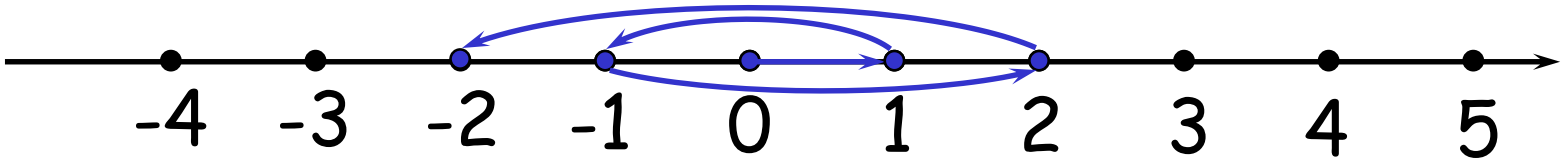
Are There More Natural Numbers or Integers?



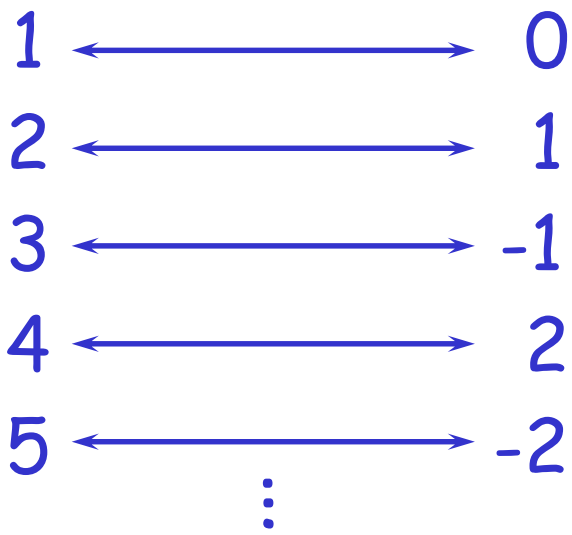
Can't be the same:
integers go to infinity in two directions!

Actually, the same.
Let's find a bijection.

Are There More Natural Numbers or Integers?

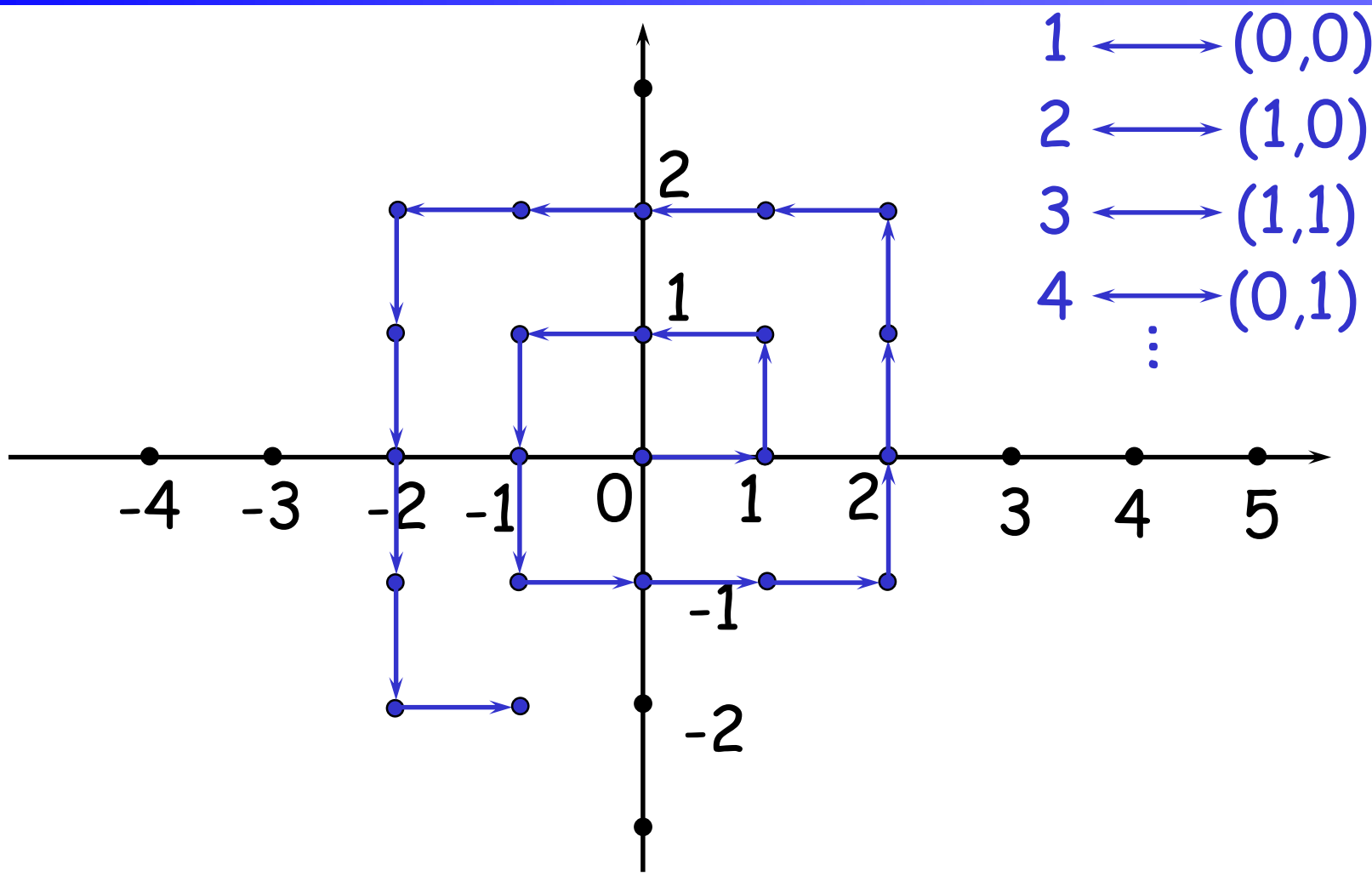


Natural numbers Integers



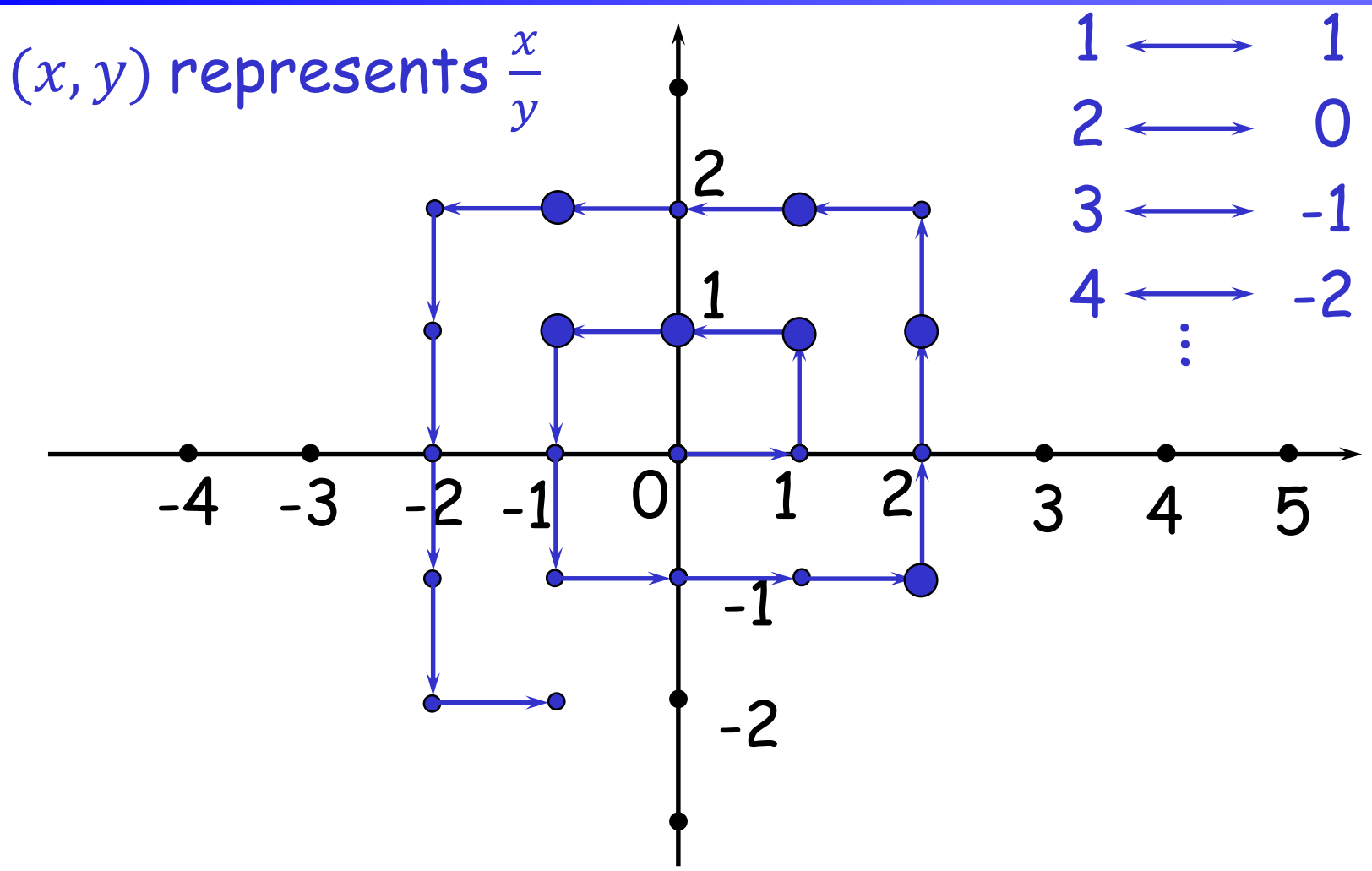
Answer: The same.

Are There More Natural Numbers or Points with Integer Coordinates on the Plane?



Answer: The same.

Are There More Natural or Rational Numbers?



Answer: The same.

Countable Sets

- Sets that have finite number of items or the same number of items as natural numbers are called countable.
- The sets of positive even numbers,

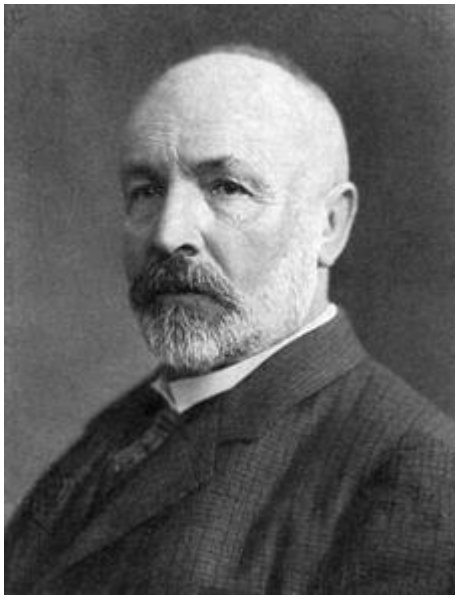
  integers,

  points on the plane with integer coordinates,

  rational numbers

  are countable.

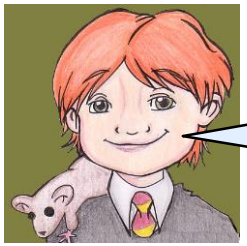
Creator of Set Theory



Georg Cantor
1845--1918

Is the Set of Real Numbers Countable?

I don't think so.
We will run out of natural numbers before we can match up all reals.



How can you be sure there isn't some clever bijection?

Cantor proved it!

The Set of Reals Between 0 and 1 is not Countable

- Suppose it is countable.
- Then there is a bijection…

Natural numbers		Reals between 0 and 1
1	⟷	0.**2**4545217 0…
2	⟷	0.4**7**2111503 7…
3	⟷	0.20**1**412041 5…
4	⟷	0.123**4**56789 0…
5	⟷	0.03131**1**3133 3…
⋮		

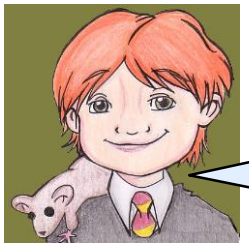
We can write a number which is different from every number on the list: 0. 1 1 0 1 0 …

It can't be a bijection!

There are more reals than natural numbers!

Are the following sets countable?

- The set of names of students in this class.
- The set of squares: 1,4,9,16,25,…
- The set of powers of two: 1,2,8,16,32,64,…
- The set of points (x, y, z) with integer coordinates.
- The set of polynomials with integer coefficients.
- The set of sets of integers.
- The set of sets of real numbers.

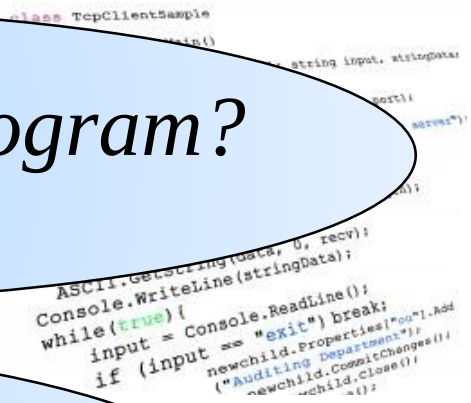


Wow! This is counterintuitive!
Can we talk about computer programs instead?

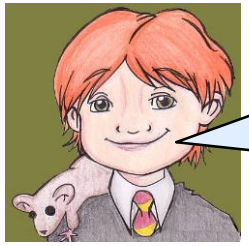
Are there more programs or problems?

Is the Set of Computer Programs Countable?

What is a computer program?



It is a set of precise instructions written in some programming language.



Can we list them like we listed rational numbers?

Nice idea!

Is the Set of Computer Programs Countable?

Yes, we can list them all:

- First, all programs of length 1.
- Second, all programs of length 2.
- Then, all programs of length 3.
- And so on...

What is a Computational Problem?

- It has an input and an output.
- The simplest problems correspond to YES/NO questions about natural numbers.

Examples:

- Input: a natural number
- Question: is it odd?
- Answer: YES/NO

Other questions: Is it prime? Is it a power of 2?...

YES/NO Problems About Natural Numbers

For each such problem, we can write correct answers on inputs 1 2 3 4 5 6 7 …

- Question: is it odd?

  YES NO YES NO YES NO YES…

- Question: Is it prime?

  NO YES YES NO YES NO YES…

- Is it a power of 2?

  YES YES NO YES NO NO NO…

How Many Problems Are There?

- Suppose there is a bijection with natural numbers

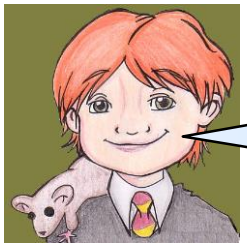
Natural numbers	Problems (with correct answers) 1	2	3	4	5	6 …
1 ⟷	YES	YES	NO	YES	NO	NO …
2 ⟷	NO	NO	YES	NO	YES	NO …
3 ⟷	YES	NO	YES	NO	YES	NO …
4 ⟷	YES	YES	YES	YES	NO	YES…
5 ⟷	NO	NO	NO	YES	NO	NO …
⋮						

We can find a problem which is different from every problem on the list: NO YES NO NO YES …

It can't be a bijection!

There are more computational problems than programs!

Not all problems can be solved by computers.



Are there any useful problems that can't be solved by computers?

The Halting problem

Given

- the source code for a program, <P>
- an input string s for P

decide whether P halts on s.

There is no program
that solves this problem!

Proof : the Halting problem is undecidable

Suppose for the sake of contradiction that program H decides the Halting problem:

On input (<P>, s),

$$H \text{ outputs } \begin{cases} \textbf{YES} \text{ if P halts on s.} \\ \textbf{NO} \text{ if it does not.} \end{cases}$$

Proof : the Halting problem is undecidable

We write a new program G that uses H as subroutine.

G=``**On input <P>,**

1. **Run H on input (<P>,<P>).**
2. **If H outputs NO, halt;**

   else enter an infinite loop.''

What does G do on input <G>?

Proof : the Halting problem is undecidable

We write a new program G that uses H as subroutine.

G=``**On input <P>,**

1. **Run H on input (<P>,<P>).**
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If H says that G halts on <G>, then G loops on <G>.

If H says that G loops on <G>, then G halts on <G>.

A contradiction!

A movie about undecidibility of Halting Problem

Proof That Computers Can't Do Everything

(The Halting Problem)

https://www.youtube.com/watch?v=92WHN-pAFCs

Summary

1. There are different kinds of infinity.
2. There are more real numbers than natural numbers.
3. There are more computational problems than programs.
4. Some really useful problems, like the Halting problem, can't be solved by computer program.