Problem of the Month.
Solutions: December 2012

MATHEMATICS

This story happened in the school of a small Scottish town. This school had exactly 1000 students. All of them had lockers that were numbered 1 to 1000. It was a very old building so probably just as you suspected there were 1000 ghosts living on premises. Every night all students were locking their lockers and every night ghosts were playing opening them.

On one night students locked their lockers as usual and went home. The night was dark and perfect for the ghosts. At 12 am sharp they got out of the walls and started playing:

- The first ghost opened all of the lockers;

- Then the second ghost locked the ones that had even numeration;

- The third ghost didn't' like it and switched it around - opened lockers (if they were locked) and locked them (if they were open) - for those that had the numbers that were divisible by 3;

- Ghost number four did the same for all lockers whose numbers were divisible by 4 (locked the ones that were open and closed the ones that were open); and so on

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As soon as the 1000th ghost finished his part the down began to break and all the ghosts had to disappear.

The students came to school in the morning as usual. How many lockers they found to be open?

Solution

The only doors that will be open in the morning are the ones whose numbers have an odd number of factors. It is due to a fact that the doors whose position was changed even number of times will remain closed and the doors whose position was changed an odd number of times will be open.

To illustrate this better consider the door number k. Suppose that $p_1 = 1, d_2, \dots, d_{m-1}, d_m = k$ are **all** the factors (in the increasing order) of the number k. Suppose m is even (i.e. number k has an even number of factors). Then the door number k will be open by the ghosts number $1, d_3, d_5, \dots, d_{m-1}$) and closed by the ghosts number $d_2, d_4, \dots, d_m$. In this case the last ghost to touch the door will close it. Thus, the door whose number has an even number of factors will remain closed.

Consider now the case where m is an odd number. Now, the door will be open by the ghosts number $1, d_3, d_5, \dots, d_m$ and closed by the ghosts number $1, d_3, d_5, \dots, d_{m-1}$. So this door will be open in the morning cause the last ghost who touched it $(ghost\ \#\ p_m)$ has opened it.

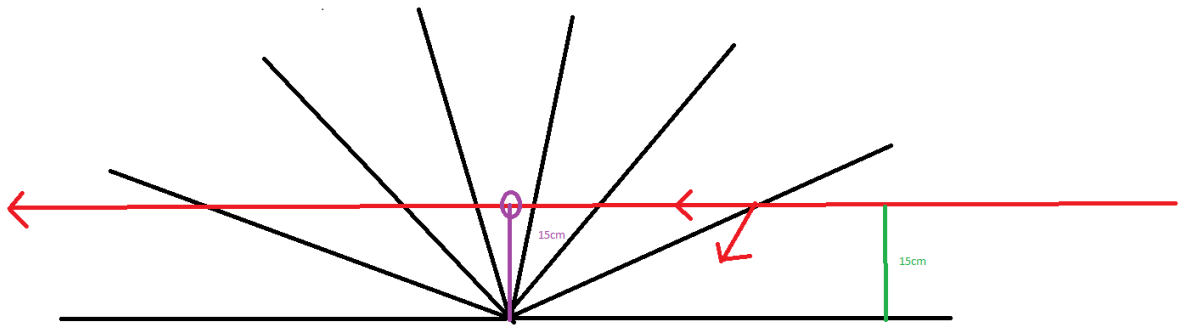
The **only** numbers that have an odd number of factors are the **perfects squares**.
That is due to a fact that for any existing factor q of a number k (not a perfect square) there exist another factor f of the same number such that $k = q * f$ (in case of a prime number k factors q and f are 1 and the number k itself). So every factor of number k in this case has a "pair" and the number of factors is even. In case of a perfects square there exist factor d of the number k for which $k = d * d$ and one of the factors will not have a "pair" thus making the number of factors odd.
There are 31 perfect squares between 1 and 1000 ($1^2 = 1$, $2^2 = 4$, $3^2 = 9$, …, $30^2 = 900$, $31^2 = 961$). So 31 lockers will remain open in the morning.

PHYSICS

Two flat mirrors are joined with their sides at an angle of 17 degrees to form a wedge, with reflective surfaces on the inside. That is, looking from above, the two mirrors simply form an angle on the plane. A laser beam parallel to one of the mirrors and passing at the distance of 15cm from it, hits the other mirror, gets reflected, hits the first mirror, gets reflected again, and so on. What would be the smallest distance between the beam and the vertex of the angle, before the beam is reflected away?

Solution *(by Andrew Khesin)*

Let us imagine the reflection of the light as not a reflection but as passing through the looking glass like Alice. There is simply another world, the mirror image of this one, on the other side. We can add the "slice" next to the old one, and the beam will continue travelling at a line parallel and 15cm away from the bottom one. We can keep adding "slices" to this "pizza" until one opens outward into infinity. The point closest to the vertex will be the one above it and since the beam is parallel it is 15cm away..



Note: Angles are more than 17 degrees. This is just an example.

CHEMISTRY

Normal saline (sometimes referred to as physiological saline or isotonic saline) solution has many medical applications. It is a 0.9% m/v solution of sodium chloride (NaCl) in water. It can be used, for example, to clean contact lenses, as a nasal spray, etc. According to Wikipedia, "home-made" saline is made by dissolving approximately half a teaspoonful of table salt into "a glass" of clean water. However, the concentration you get according to this procedure is not too accurate. Imagine, you need to prepare exactly 0.9% m/v saline solution, and you have no standard spoons, cups or "glasses" (or course, your kitchen scales appeared to be broken too). The only things you have are: your brain, Internet, salt, water, and some (non-standard) cups, glasses, pans. Please, propose the procedure for preparation of normal (0.9% m/v) saline with maximal possible accuracy.

Solution

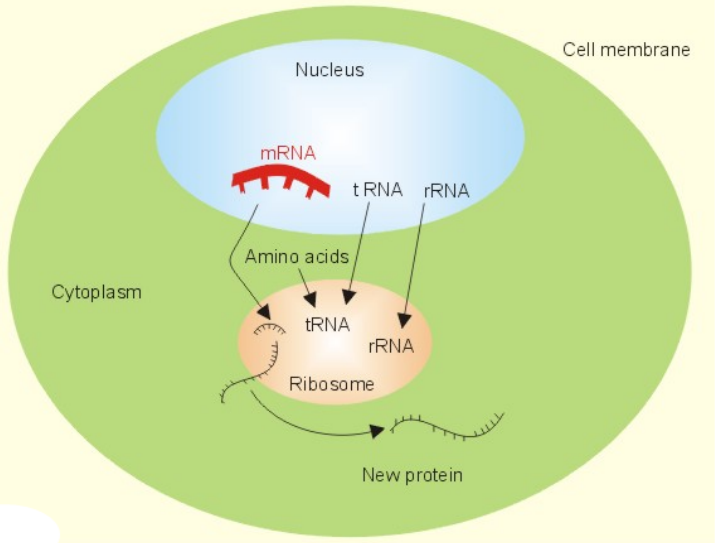
Solubility of sodium chloride (table salt) is 318 g/L. That means that when you add so much salt that it doesn't dissolve in water any more, 1 liter of such a solution contains 318 grams of salt; solubility of NaCl is almost independent of temperature (so the temperature in your kitchen doesn't matter).

Prepare saturated solution of table salt; to do that, just add salt gradually to water (with stirring) until it stops dissolving. After that, add some extra of salt, let the solution sit for several hours, shaking it from time to time. If some amount of salt remains not dissolved, the solution is "saturated", and its concentration (the largest possible concentration of salt in water) is 318 g/L. This is an exact concentration, which we obtained without using measuring glasses or scales. 318 grams per liter is equal to 31.8 g per 100 milliliters, i.e. 31.8 %.

Now we can prepare any concentration we want simply by dilution of the solution obtained, the only thing we need is to calculate dilution. If we pour 9 full glasses of water to the pan, and add one glass of the saturated salt solution, we will get 31.8/(1+9)=3.18% solution. Since we need 0.9% solution, we have to dilute our 3.18% solution again. If we take two cups of this solution, add it to the second pan, and, using the same cup, add 5 cups of water, the concentration will drop 3.5 times, and will be equal to 3.18*2/(2+5)=0.901%, which is very close to what we need.

BIOLOGY

It is widely accepted that in eukaryotes protein synthesis takes place in the cytoplasm.



Why do you think this is the case? (Wouldn't it be easier and faster to synthesize at least nuclear proteins "on site" without evolving elaborate mRNA– and protein-transporting mechanisms?)

Interestingly enough, almost all the components of the protein synthesis machinery can be found in eukaryotic nuclei making "nuclear translation" a plausible hypothesis. Please propose as many potential functions of nuclear translation as possible (at least two, or more for extra credits).

Solution

The nucleus is the principal defining feature of eukaryotic cells. The genetic material of the cell is stored in the nucleus and is transcribed into messenger RNAs (mRNAs), which are then processed and exported to the cytoplasm. So the orthodoxy goes, once in the cytoplasm mRNAs are "read" by rotund factories called ribosomes and are translated into proteins. That transcription and translation take place in two different cellular compartments distinguishes eukaryotic cells from bacteria, which do not have a nucleus. This spatial separation protects cells from the deleterious effects of making faulty proteins, which could happen if incompletely processed mRNAs were to be translated in the nucleus. It also allows more time for post-transcriptional processing of the mRNA molecules before translating them into functional proteins. These two features are more important to complex (compared to bacteria) eukaryotic cells where every mistake may be harmful to the whole organism.

In contrast, in prokaryotes translation is coupled to transcription: translation of the new RNA molecule starts before transcription is finished. Here, the gene expression machinery goes for speed instead of the quality control because even the deleterious mutations are likely to be tolerated - a single cell with some disadvantage will be easily outgrown

(hence, eliminated from the population) by “normal” bacteria.

Translation in the nucleus was first described almost 70 years ago by Allfrey (1954), who reported a rapid incorporation of radioactive amino acids into nuclear proteins. The controversy it raised - the nucleus can be an active site of protein synthesis - still remains unresolved. The most fundamental question here is what would be the function of such a process? There are 3 possible scenarios which could potentially explain observations discussed above.

1. Nuclear translation is used to synthesize nuclear proteins. There have been studies of substantial buildup of newly synthesized proteins in the nucleus and intact poly-ribosomes have been detected there.
2. Nuclear translation plays a role in mRNA quality control. A number of recent publications discussed a process in which ribosomes scan for mRNAs with premature stop codons and mark them for degradation. This process (called Nonsense- Mediated Decay) may take place in the nucleus.
3. Nuclear translation may be non-specific and non-functional, e.g. noise in the system where the right components got assembled together in some random fashion.

It is not yet clear which scenario (or combination of scenarios) plays a more prominent role in this process. Another missing piece of the puzzle is the identity of the (presumably) mRNAs that serve as substrates for nuclear translation. All these questions are under active investigation now.