

PROBLEM OF THE MONTH



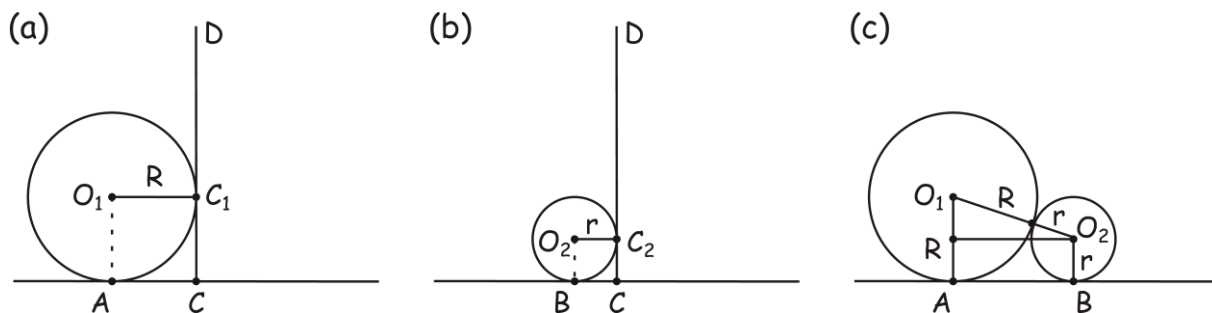
February 2013

MATHEMATICS

Two spheres, which are tangent to each other, are also tangent to a plane and to a line perpendicular to that plane.

What is the ratio of the spheres' radii if the points where the spheres touch the plane form:

- a right triangle with the point at which the line intersects the plane?
- an isosceles triangle with the point at which the line intersects the plane?



Solution.

Let us draw three cross-sections as shown in the Figure above. In all three cases the plane of the cross-section is perpendicular to the horizontal plane to which the spheres are tangent.

One passes through line CD, which is perpendicular to this plane and to which both spheres are also tangent, and the center O_1 of the larger sphere, (a), the other passes through line CD and the center O_2 of the smaller sphere, (b), and the third plane passes through centers, O_1 and O_2 , of the two spheres, (c). Triangle ABC is formed by points A and B where the spheres touch the plane, and point C where line CD crosses the plane. It is clear from the Figure that

$$|AC| = R, |BC| = r, \text{ and } |AB|^2 = (R+r)^2 - (R-r)^2 = 4Rr.$$

Here we used the fact that line tangent to the circle is perpendicular to the radius to the point of tangency, so ACC_1O_1 , BCC_2O_2 and ABO_2R are all rectangles. It is easier to first answer the second question of the problem.

(b) If triangle ABC is isosceles, then $|AC|^2 = |AB|^2$, or, $R^2 = 4Rr$, so $R = 4r$, $R/r = 4$.

(a) If ABC is a right triangle, then it must satisfy Pythagorean theorem. If the hypotenuse is $|AC| = R$, then we have $|AC|^2 = |AB|^2 + |BC|^2$, or, $R^2 = 4Rr + r^2$. Dividing this equation by r^2 and denoting the ratio $R/r = x$, we obtain the square equation for the ratio x , $x^2 - 4x - 1 = 0$. This

equation has two solutions, of which only one is positive, $x = 2 + \sqrt{5}$. If the hypotenuse is $|AB| = 2\sqrt{Rr}$, we similarly have $4Rr = R^2 + r^2$, and obtain a square equation, $x^2 - 4x + 1 = 0$.

This equation also has two solutions, $x = 2 \pm \sqrt{3}$, of which one is the inverse of the other. Since

we denoted $x = R/r > 1$, we choose $x = 2 + \sqrt{3}$. Solution with the minus would correspond to the case $R < r$. Hence, the answer is: $x = 2 + \sqrt{3}$, or $x = 2 + \sqrt{5}$.

PHYSICS

A year on the planet X lasts exactly 300 days. Any day on its equator lasts exactly 24 hours.

What could be the period of the planet's rotation about its axis (in seconds)?

Note: a day is defined as a time between two sunsets.

Solution:

Imagine that the planet X does not spin at all, and only orbits its “Sun”. From the point of view of a person on the planet, the “Sun” moves around the planet, making exactly one turn every year. If we now assume that the planet does spin, and making N per year, from the point of view of a person on the planet the “Sun” makes either N+1 or N-1 turns, depending on the direction of planet spinning.

Therefore, the actual number of turns the planet X makes about its axis every year is not 300 (number of days), but rather 301 or 299. The difference of 1 comes from the orbital motion. We can now find the period by calculating the length of the year (in seconds): $300 \times 24 \times 3600s$, and dividing it by the total number of turns (301 or 299, respectively):

$$T = \frac{300}{300 \pm 1} \times 24 \times 3600s = 86113s \quad \text{or} \quad 86689s$$

BIOLOGY

"We are of the same blood you and I"- told Mowgli in Kipling's "Jungle Book". In some cases this phrase may not only uphold health, but also save lives.

People differ from each other in blood type the same way that they differ in hair and eye color. It is of utmost importance in the flow of life, does not take into consideration nationality, gender, or age, is handed down genetically, and lends itself to an individual set of characteristics for each of us.

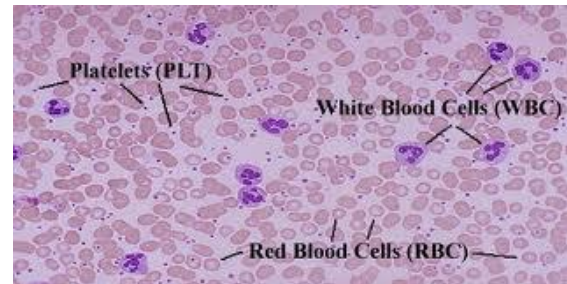
In modern times, four blood types are known to mankind, which came about in the process of evolution: O, A, B, and AB. The most common type is O, while the least common is AB. Is there a significant difference between them? We'll try to figure it out.

1. In what ways do the blood types differ from one another?
2. Why can the O type be transfused to everyone? What other factor is crucial to consider while selecting a blood donor?
3. At a maternity ward in a hospital, two girls were mixed up. One of them was found to have an O blood type, while the other has AB. One set of parents has the blood types O and A, while the other has A and B. Match the daughters to the parents.

Solution.

1. Blood is a mixture of cells and watery liquid, called plasma that the cells float in. It also contains other things like nutrients (such as sugar), hormones, clotting agents, and waste products to be flushed out of the body.

There are three kinds of cells in the blood: red blood cells, white blood cells, and platelets. Red blood cells carry oxygen to the body's tissues and remove carbon dioxide. White blood cells help defend the body against infection by producing antibodies, which help destroy foreign germs in the body. Platelets, the smallest blood cells, help to clot the blood and control bleeding. Plasma is the liquid part of whole blood and contains a mixture of water, proteins, electrolytes, carbohydrates, cholesterol, hormones, and vitamins.



Not all blood is the same. The differences in human blood are due to the presence or absence of certain protein molecules (antigens) on the surface of red blood cells. If a person does not have these proteins in their blood, they may make antibody molecules to attack them.

O-	A-	B-	AB-
O+	A+	B+	AB+

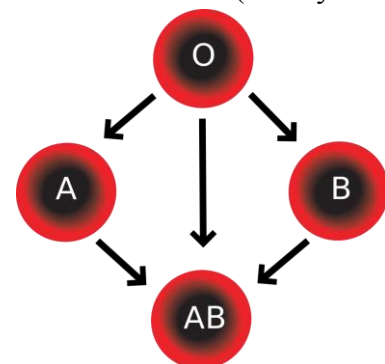
Using the ABO blood classification system, every human being belongs to one of four blood types: A, B, AB, or O and positive or negative Ph factor.

Your ABO blood type is based on the presence or absence of the A and B antigens on your red blood cells. The A blood type has only the A antigen and the B blood type has only the B antigen. The AB blood type has both A and B antigens, and the O blood type has neither A nor B antigen.

The Rhesus factor, also known as the Rh factor, is another antigen that is found on the covering of the red blood cells. Most people have the Rhesus factor and they are considered to have a "positive" (+) blood type, such as A+ or B+. Those who don't are considered to have a "negative" (-) blood type, such as "O-" or "AB-." ABO types and Ph can be combined into 8 possible blood types:

By the time you are six months old, you naturally develop antibodies against the antigens your red blood cells lack. For instance, a person with A blood type will have anti-B antibodies, and a person with B blood type will have anti-A antibodies. Type AB blood contains both "A" and "B" proteins so these individuals do not make any antibody against "A" or "B" (if they did, they'd attack their own cells.) Type O blood has neither "A" nor "B" protein present so individuals with O blood make antibodies against both "A" and "B" proteins.

2. If you receive a blood transfusion, you must receive blood that is compatible to yours. If you have type A blood, you cannot receive B blood because your body's anti-B antibodies will fight the B blood's B antigens. If blood from different groups is mixed together, the antibodies make the red blood cells stick together. Large clots form and can block blood vessels and it can cause many health problems and even death.



In terms of blood transfusions, the rule is to give the same blood type or a blood type lacking the

antigen against which the individual has the corresponding antibody in the plasma. Therefore, type A individuals can receive blood from type A or type O individuals; type B can receive blood from type B or type O individuals; type AB can receive blood from type AB, type A, type B or type O individuals; and type O can receive blood only from type O individuals.

People with type AB blood are called universal recipients, because they can receive any of the ABO types. People with type O blood are called universal donors, because their blood can be given to people with any of the ABO types.

Just like eye or hair color, our blood type is inherited from our parents. Each biological parent donates one of two ABO genes to their child. The A and B genes are dominant and the O gene is recessive. For example, if an O gene is paired with an A gene, the blood type will be A.

3. When at a maternity ward in a hospital, two girls were mixed up, one of them was found to have an O blood type, while the other has AB.

One set of parents has the blood types O and A. So their daughter can have either O or A blood type. And it means that the girl with AB blood type cannot be their daughter, but the girl with O blood type can be.

Another set of parents has the blood types A and B. So their daughter can have any blood type and the girl with AB blood type is their daughter.

		Father's Blood Type				
		A	B	AB	O	
Mother's Blood Type	A	A or O	A, B, AB, or O	A, B, or AB	A or O	Child's Blood Type
	B	A, B, AB, or O	B or O	A, B, or AB	B or O	
	AB	A, B, or AB	A, B, or AB	A, B, or AB	A or B	
	O	A or O	B or O	A or B	O	

Answer: The first girl - the daughter of the first pair of parents, the other girl - the daughter of the second set of parents.

CHEMISTRY

A major qualitative characteristic of solutions is their **concentration**.

Concentration is a number that shows the amount of a dissolved substance (a solute) per 1 liter (or one kilogram, one gallon, etc) of the solution. Sometimes, a concentration is measured in percents. For example, a 10% aqueous solution of NaCl (table salt) contains 10 grams of NaCl per 100 grams of the *solution* (not per 100 grams of water).

In my lab, I need to prepare the 8% solution of sulfuric acid (H_2SO_4). I found 30% and 5% solutions of H_2SO_4 on my shelf. In which ratio should I mix them to obtain a 8% solution?

Solution

Of course, we can solve this problem using simple algebra. However, that is not too interesting. Let's see if some general solution can be obtained.

Actually, this type problem is very common. For example, winemakers have to solve similar problem frequently when they are blending different wines. Therefore, it is interesting and important to have some general solution.

Suppose you have two solutions of some substance X. The first has a concentration A, and the second solution has a concentration B. You take M_a grams of the solution A and M_b grams of the solution B. As a result, you get a solution C. Obviously, there is $A \cdot M_a$ grams of X in the first solution, and $B \cdot M_b$ grams of X in the second solution. When we mix them together, the total amount of X remains the same, so the solution C contains $C \cdot (M_a + M_b)$ of the substance X. The same idea can be said in a mathematical way as follows:

$$A \cdot M_a + B \cdot M_b = C(M_a + M_b)$$

This equation can be transformed as follows:

$$A \cdot M_a + B \cdot M_b = C \cdot M_a + C \cdot M_b$$

and then as follows:

$$(A - C)M_a = (C - B)M_b$$

Obviously, we get a simple proportion, which transforms to:

$$(A - C)/(C - B) = M_b/M_a$$

What does it mean? Translated from the mathematical language to English, that means to obtain a final solution having a concentration C, you need to take solution A - C parts of the solution B and C - B parts of the solution A. Obviously, to get non-negative amounts, A should always have higher concentration than B.

This conclusion can be graphically represented as so called a **Pearson's square**:

A (concentration of the first solution)

C - B (amount of the solution A)

C (desirable concentration)

B (concentration of the first solution)

A - C (amount of the solution B)

In our case, $A = 30$, $B = 5$, $C = 8$. Accordingly, the Pearson's square is:

30

8 - 5 = 3

8

5

30 - 8 = 22

The answer is: 3 to 22.

COMPUTER SCIENCE

Write a Java program that takes 10 numbers as user input and outputs them in increasing order. Java library sort methods can not be used in your program.

Solution:

```
import java.util.*;
import java.io.*;
class FebPoM
{
    public static void main(String[] args) throws IOException
    {
        double[] numbersIn = new double[10];
        System.out.println("Please enter 10 numbers: ");
        Scanner input = new Scanner(System.in);
        for (int i = 0; i < 10 ; i++)
        {
            numbersIn[i] = input.nextDouble();
        }
        System.out.println("Unsorted numbers: " + Arrays.toString(numbersIn));
        insertionSort(numbersIn);
        System.out.println("Sorted numbers: " + Arrays.toString(numbersIn));
    }
    public static void insertionSort(double [] a ){
        for(int i=1; i< a.length; i++){
            double temp = a[i];
            int j;
            for (j = i-1; j>=0 && temp < a[j]; j--){
                a[j+1] = a[j];
            }
            a[j+1] = temp;
        }
    }
}
```