PROBLEM OF THE MONTH

January 2013

MATHEMATICS

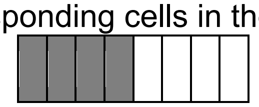
Alice wrote a secret letter on a piece of paper and put it in one of the 8 numbered envelopes. Bob has to find out, which envelope contains the letter by asking Alice questions to which she can answer only “Yes” or “No”. Bob knows that Alice’s answers are not necessarily truthful, but he also knows that she would not lie more than once.
1. What do you think is the minimum amount of questions Bob has to ask to find the envelope with the letter?
2. What would be the smallest amount of questions Bob has to ask if there would be not 8 but 250 envelopes?
You do not need to PROVE that Bob’s strategy utilizes the smallest number of questions, but you should prove that the strategy WORKS (i.e. Bob should know for certain where Alice put her letter after asking his series of questions).

Solution

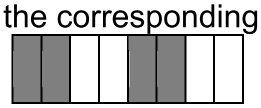
This problem can be solved in different ways, and we shall outline two slightly different solutions below. In both cases we make use of the following observation: If, after asking a number of questions, Bob asks Alice “Have you lied answering one of the questions so far?” the answer Alice gives will be truthful in the following sense. If Alice answers “yes,” then she indeed lied in answering one of the previous questions – including the last one, i.e. she perhaps lied when she answered “yes” to this last question. If she answers “no”, then she has not lied, as she cannot lie twice. Keeping this observation in mind, Bob can use one of the following two algorithms.

A. Solution by decimation (binary search). Bob asks questions in the following sequence.

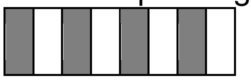
1. Does the envelope with your letter belong to the first half, i.e. is it one of the envelopes numbered from 1 to 4? Let us assume that the answer is “No”, and let us shade the corresponding cells in the table below.



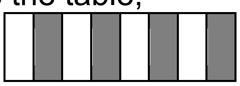
2. Is the letter in the first half of either of the halves? Let us assume the answer is “No” and shade the corresponding cells.



3. Is the letter in the first half of either of the quarters? Let us assume the answer is “No” and shade the corresponding cells.



4. At this point, if all of Alice’s answers have been truthful, the letter must be in the envelope that has never been shaded, i.e. the envelope #8 in the present example. However, knowing that one of the Alice’s answers could have been incorrect, Bob has to ask Alice whether she lied in answering one of the questions so far. As we have noted at the beginning, if Alice answers “No”, then the problem is solved after four questions. If, however, Alice answers “Yes”, then Bob has to figure out which answer was incorrect, and then reverse the shading in the corresponding table. Bob does this by repeating the decimation as above. Since Alice has already lied once, her answers will be truthful. Hence, Bob needs to ask two more questions.
5. “Did you lie answering one of the first two questions?” – Suppose the answer is “No”. Then, Alice must have lied answering one of the last two questions. Therefore, Bob asks,
6. “Did you lie answering the third question?” Suppose, the answer is “Yes”. Then, Bob re-shades the table,



And the letter is in the envelope #7, which Bob knows after asking 6 questions.

B. Solution using the binary number system. Bob notes that binary notation for numbers from 0 to 7 has three bits, i.e. such number can be written as $\boldsymbol{a}*2^0+\boldsymbol{b}*2^1+\boldsymbol{c}2^2$, where ***a, b,*** and ***c*** are bits, i.e. can be either 0 or 1. Therefore, he numbers envelopes from 0 to 7 using binary notation ***abc***, and aims at figuring out what are the three bits encoding the number of the envelope which contains the letter. He asks the following questions.
1. “If I write the number, 0 to 7,of the envelope with the letter using binary notation, is the first bit zero?” – Suppose the answer is “No”.
2. “Is the second bit zero?” – Suppose the answer is “No”.
3. “Is the third bit zero?” – Suppose the answer is “No”.
4. As in the previous case, if all of Alice’s answers have been truthful, the letter must be in the envelope #7 (remember, now they are numbered from 0 to 7 – this is the same as #8 in the previous solution, where envelopes were numbered from 1 through 8). However, knowing that one of the Alice’s answers could have been incorrect, Bob has to ask Alice whether she lied in answering one of the questions so far. As we have noted at the beginning, if Alice answers “No”, then the problem is solved after four questions. If, however, Alice answers “Yes”, then Bob has to figure out which answer was incorrect, and then reverse the corresponding bit. To figure out which of the four questions asked so far was answered incorrectly, Bob can number these questions using the binary notation, ***ab***, and then figure out what are the two bits in that number is by asking the following two questions.
5. “If I write the number of the question that has been answered incorrectly using binary notation, is the first bit zero?”
6. “Is the second bit zero?” At this point Bob knows which bit to reverse in the envelope’s number, and therefore knows where the letter is upon asking 6 questions.

Solution for the case of 250 envelopes proceeds in a similar manner. In order to apply decimation, we add 6 more empty envelopes, so that the total number of envelopes becomes 2^8 = 256. Bob asks 7 questions,

1. Does the envelope with your letter belong to the first half?
2. Does the envelope belong to the first half of either of the halves considered in the previous question?
3. Does the envelope belong to the first half of either of the quarters considered in the previous question?
4. Does the envelope belong to the first half of either of the 1/8's considered in the previous question?
5. Does the envelope belong to the first half of either of the 1/16's considered in the previous question?
6. Does the envelope belong to the first half of either of the 1/32's considered in the previous question?
7. Does the envelope belong to the first half of either of the 1/128's considered in the previous question?
8. At this point, Bob has to ask Alice whether she lied in answering one of the questions so far. In the worst-case scenario she answers "Yes", and Bob needs to figure out which of the 8 questions was answered incorrectly. Using decimation again, this requires 3 more questions. After that, Bob has to ask the last question #12, corresponding to the final decimation, and equivalent to simply asking "Is envelope with the letter numbered even, or odd?" Using this algorithm, Bob finds out which envelope contained the letter after asking 12 questions.

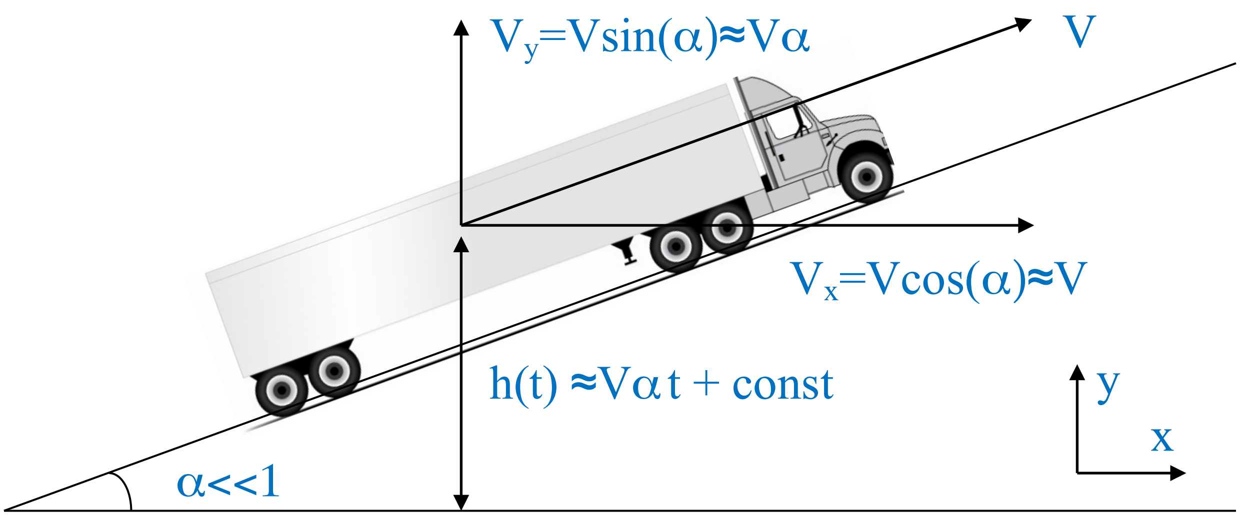
Similarly, using 12 questions Bob can find out the number of the envelope with the letter using its binary notation. Now, the number can range from 0 to 249, and therefore requires 8 bits, ***abcdefgh***, in binary notation. In this case too, it is important that Bob asks Alice whether she lied in answering one of the questions so far after asking 7 questions - this optimizes the algorithm of finding out which question has been answered incorrectly by making the total number of questions (8) a power of 2.

PHYSICS

Joe the trucker is delivering 48000 lbs of cargo in his rig equipped with a 550 hp engine. Joe is trying his best to keep the truck's speed at the 55 mph limit. On an uphill climb Joe noticed that he could only maintain this speed up to a 3% grade[1], and (at this grade and 55 mph) the truck fuel efficiency gauge was measuring 2.2 mpg (miles per gallon). Estimate the fuel efficiency Joe was getting on a flat portion of his route, if the mass of his truck without cargo is 32000 lbs.

Solution

As is true for any heavy vehicle, climbing uphill requires a lot more engine power than cruising on a flat road at the same speed. This is because, in addition to the engine power that counteracts various energy losses common for both situations (such as air resistance, tire friction, etc.), a significant power is required to keep increasing the potential energy of the vehicle, or, simply speaking, to keep lifting the truck against the gravity force.



The potential energy of the truck (of mass m including the cargo of mass M) could be written as

$$W=(m+M)g\, h(t), \tag{1}$$

where $g=9.81$ m/s^2 and h(t) is the time-varying height of the truck's center of mass with respect to some fixed reference. The change of W per unit time (and the engine power P_1 causing this change) is then

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[1] % grade is a common measure of road slope in the US. This is what's displayed in street signs, driver's navigation aids, etc. It is defined as $100 * vertical_{elevation}/horizontal_{run}$, so, for instance, a 45-degree slope is 100% grade.

$$P_1=(m+M)g\ V_y, \tag{2}$$

where V_y stands for the vertical component of the truck velocity, which is, of course, the same as the change of h(t) per unit time.

When moving on a linear slope with the total velocity V,

$$V_y=V\sin(\alpha). \tag{3}$$

Note that for small angles, α <<1, that we typically encounter on the road,

$$\sin(\alpha)=\tan(\alpha)=\alpha, \tag{4}$$

where α is the incline angle, expressed in radians. To convert a small angle expressed in % grade into radians we can simply divide by 100, i.e. 3% grade is equivalent to α=0.03. (For large angles one must, however, use α[radians]=ArcTan(α[% grade]/100).)

The problem states that the trucker couldn't maintain the speed when going uphill for grades higher than 3%. We infer that, at the given incline and speed, the engine runs at its maximum power, so

$$P_{max}=P_0+P_1, \tag{5}$$

where P_0 is the engine power required to ride at the same speed on a flat route. Substituting Eq. (2) and making use of Eqs. (3) and (4), we get

$$P_0/P_{max}=1-P_1/P_{max}=1-(m+M)g\ V\ \alpha/\ P_{max}\ . \tag{6}$$

Assuming that the fuel consumption is directly proportional to the power produced by the engine, we conclude that the fuel efficiency (sometimes also called fuel economy) is inversely proportional to the engine power, so we finally get

$$\text{mpg\_flat} = \text{mpg\_grade}\times P_{max}/P_o = \text{mpg\_grade}/[1-(m+M)g\ V\ \alpha/\ P_{max}\ ], \tag{7}$$

where mpg_grade=2.2 mpg is the fuel efficiency going uphill, and mpg_flat is that on a flat road.

Now it's time to substitute the numerical values while keeping track of the units. While using other systems is definitely possible, it is most convenient to convert everything to the metric system. For this problem this amounts to converting horse power to watts, miles per hour to meters per second, and pounds to kilograms. Consulting any of the easily available references (i.e. Wikipedia), we get

$$P_{max} = 550\text{ hp} = 410135\text{ W} \tag{8a}$$

V = 55 mph = 24.5872 m/s (8b)

$m+M$ = 32000 lbs + 48000 lbs = 80000 lbs = 36287.4 kg (8c)

Substituting these in (7) we get the final answer

mpg_flat = 2.78 mpg_grade ≈ 6 mpg, (9)

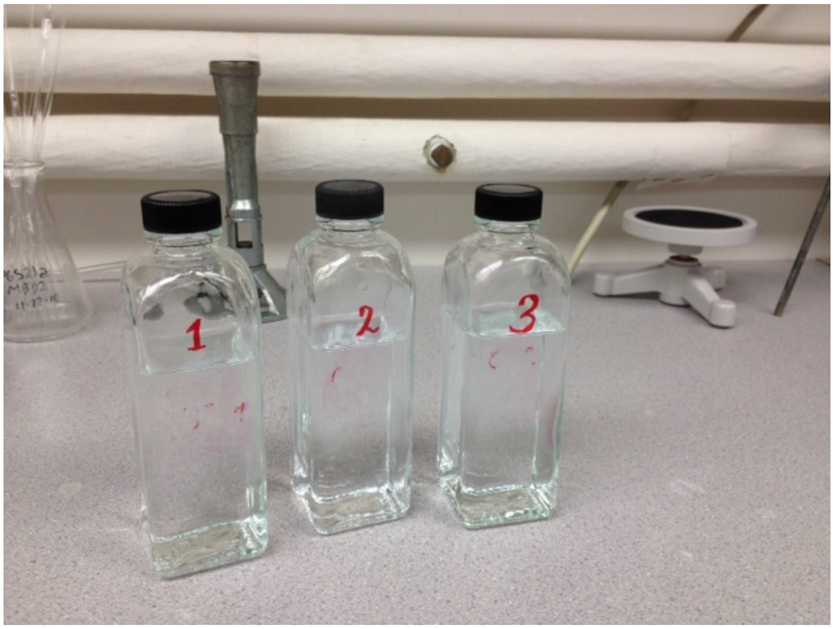
that is the fuel efficiency on the flat portion of the route was approximately 6 miles per gallon.

Note that when substituting numerical values into Eq. (7) we could have computed the final answer to many decimal places. Keeping them, however, would incorrectly overstate the accuracy of the answer, which should never exceed the accuracy of the assumptions and approximations we made, explicitly or implicitly. Listing all of these assumptions and discussing their accuracy is beyond the scope of this solution. Still, can you think of at least five of them?

CHEMISTRY

In my lab, I prepared three water solutions, sodium sulfate, calcium hydroxide (lime water), and calcium chloride, and labeled them (fig 1).

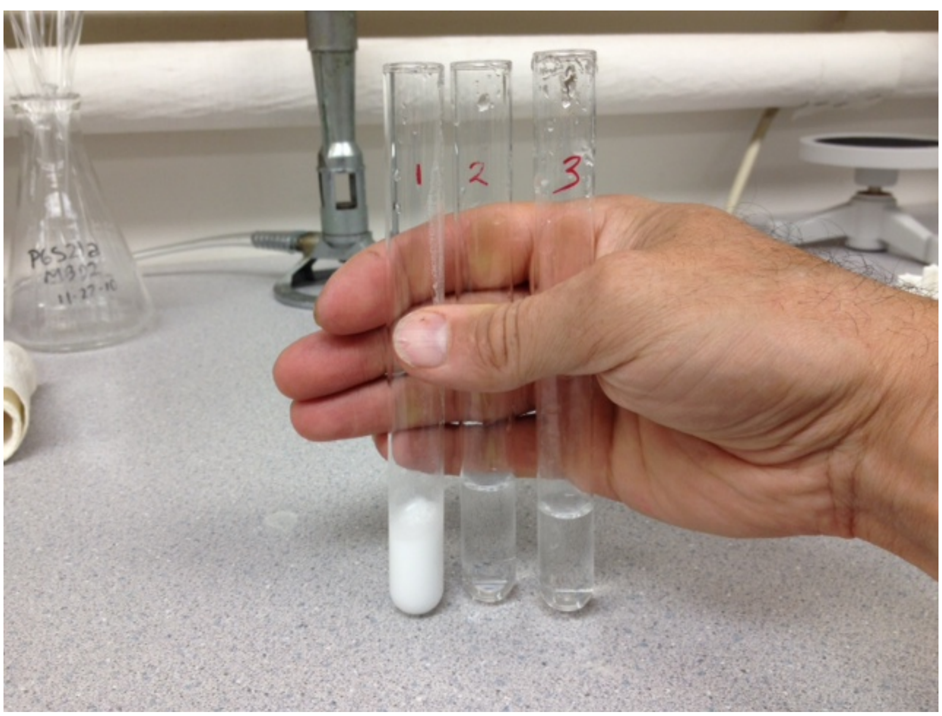
When I came to the lab next morning, I found someone had wiped out all the labels (by accident). According to the rules, I had to dispose all those solutions. However, before doing that, I decided to experiment a little bit. I decided to check if I am able to restore the labels. Firstly, I marked these bottles as “1”, “2”, and “3” (because I didn’t know the actual content of each of them, fig 2),



and poured small amount of each solution into the test tubes marked accordingly (“1”, “2”, and “3”). Then I added few drops of phenolphtalein solution to each tube (fig 3), and that is what I observed:

Then I washed all three test tubes, and poured fresh solutions “1”, “2”, or “3” to each of them. As previously, the number on the bottle corresponded to the test tube number. After that, I added a solution “1” to the solutions “2” and “3”, and I observed formation of precipitates in the tube “2”, and “3”:

Then I washed test tubes again, and added the solution “3” to the solutions “1” and “2”. Only a solution “1” gave a precipitate with a solution “3”; no precipitation was observed in the test tube “2”:



After that, I concluded I am able to restore all three labels. Am I right? Is it really possible to restore the labels, and if yes, which labels ($Ca(OH)_2$, Na_2SO_4, or $CaCl_2$) correspond to the solutions "1", "2", and "3"?
Please, draw the equations of each reaction if you can. As usually, all information you need to solve this problem is available at our SchoolNova web site (http://schoolnova.org/nova/classinfo?class_id=chemistry101&sem_id=f2013 or http://schoolnova.org/nova/classinfo?class_id=chemistry101&sem_id=s2014).

Solution

Of course, I was able to restore the labels. My considerations were as follows.
Since sodium sulfate and calcium chloride solutions cannot be basic (both of them are formed by a strong acid and strong a base), the only solution that becomes pink after addition of phenolphtalein is calcium hydroxide. Therefore, the experiment 1 demonstrated the bottle 2 contains calcium hydroxide.
Now we know that one of the bottles #1 and #3 contain sodium sulfate, and another one contains calcium chloride. Obviously, by mixing them together we get a white precipitate of calcium sulfate (gypsum):

$$CaCl_2 + Na_2SO_4 = CaSO_4\,(solid) + 2\,NaCl$$

However, this experiment gives us no additional information: we still do cannot determine if sodium sulfate is in the bottle #1 or in the bottle #3 (obviously, calcium chloride will be in another bottle). To determine that, we need some third chemical that reacts with one solution and does not react with another. Fortunately, we have such a reagent. It is calcium hydroxide (a solution #2), which yields gypsum when mixed with sodium sulfate:

$$Ca(OH)_2 + Na_2SO_4 = CaSO_4\,(solid) + 2\,NaOH$$

and gives nothing when mixed with calcium chloride. Therefore, since the solution #2 gives precipitation with the solution #1, and no precipitation takes place with the solution #3, sodium sulfate is in the bottle #1.
To summarize, the answer is:

1st bottle - sodium sulfate
2nd bottle - calcium hydroxide
3rd bottle - calcium chloride

BIOLOGY

The father of modern genetics, Gregor Mendel studied the inheritance of traits in pea plants. Please write a concise scientific report of his groundbreaking work answering the following questions: a) what was Mendel's main hypothesis? b) Why did he choose pea as an object? c) Did he consider anything else and would it be possible to use another species for his work? d) How did he perform his experiments? e) What did he observe? F) How did he interpret his results?

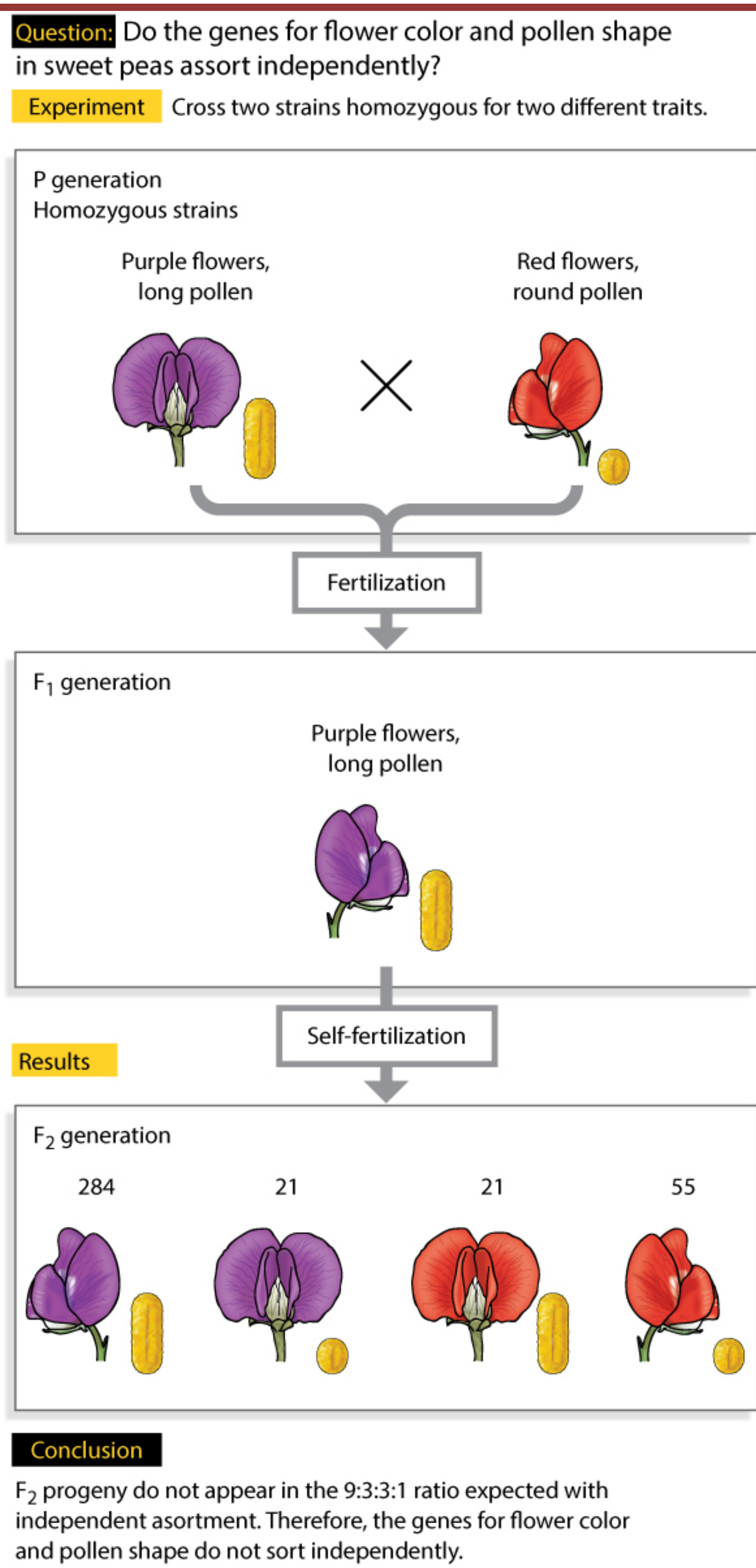
In his work Mendel described 7 monogenic traits. It is often said that he got extremely lucky because pea plants have exactly 7 pairs of chromosomes thus all these traits were not linked. Yet, later experiments revealed that 7 traits studied by Mendel are located on 5 chromosomes. Why do you think these traits are not linked?

Solution

The answer to the first part of the problem is well described in a number of books and on-line, thus here we provide a detailed answer to the second part only.
Although Gregor Mendel developed his groundbreaking principles of inheritance in the mid-1800s, the importance of his work went largely unnoticed by the scientific community until the early 1900s. At that time, numerous researchers began to conduct experiments that upheld many of Mendel's ideas; however, they also discovered several situations that represented apparent deviations from these principles.
In 1905, William Bateson, Edith Rebecca Saunders, and Reginald Punnett were examining flower color and pollen shape in sweet pea plants by performing di-hybrid crosses similar to those carried out by Mendel. In particular, these researchers crossed homozygous pea plants that had purple flowers and long pollen grains with homozygous plants that had red flowers and round pollen grains. Prior to the cross, the trio noted that purple flowers (P) were dominant over red flowers (p), and that long pollen grains (L) were dominant over round pollen grains (l). The F_1 generation of plants resulting from the PPLL x ppll cross was therefore doubly heterozygous (PpLl), and all of the F_1 plants had purple flowers and long pollen grains. Next, Bateson, Saunders, and Punnett decided to cross the F_1 plants with each other. After this cross, the researchers expected the F_2 generation to have a 9:3:3:1 ratio (nine plants with purple flowers and long pollen grains, to three plants with purple flowers and round pollen grains, to three plants with red flowers and long pollen grains, to one plant with red flowers and round pollen grains). Instead, they observed the results shown in Table 1 and Figure1, and these results were found to be statistically significant.

Table 1: Characteristics of the F_2 generation

Phenotype	**Expected**	**Observed**
Purple, long	**1199**	**1528**
Purple, round	**400**	**106**
Red, long	**400**	**117**
Red, round	**133**	**381**
Total	**2132**	**2132**



As Table 1 indicates, Bateson, Saunders, and Punnett observed that their crosses produced a deviation from the predicted Mendelian independent assortment ratios. During their analysis, the researchers realized that there was an excess in the number of parental phenotypes (purple-long and red-round) in the F_2 results. In particular, of the 2,132 F_2 plants, 1,199 were expected to have purple flowers and long pollen grains, but instead, there were a whopping 1,528 plants with this phenotype. Similarly, only 133 plants were expected to have red flowers and round pollen grains, but the researchers observed nearly three times that many (381). It is now understood that the differences between the expected and observed results were statistically significant ($P < 0.005$), which means that the data could not be explained solely by chance. Because the parental phenotypes reappeared more frequently than expected, the three researchers hypothesized that there was a coupling, or connection, between the parental alleles for flower color and pollen grain shape, and that this coupling resulted in the observed deviation from independent assortment. Indeed, Figure 1 shows an example of a cross between homozygous pea plants with purple flowers and long pollen grains and homozygous plants with red flowers and round pollen grains that exhibits linkage of the parental alleles.

But why are certain alleles linked? Bateson, Saunders, and Punnett weren't sure. In fact, it was not until the later work of geneticist Thomas Hunt Morgan that this coupling, or linkage, could be fully explained.

Why didn’t Mendel observe genetic linkage? So if linkage exists, why didn't Mendel detect it while carrying out his crosses in pea plants? In part, this was the case because Mendel studied seven genes, and the pea plant has seven chromosomes. Still, Mendel didn't choose pairs of genes that were always on different chromosomes; some of the pairs of genes that Mendel studied were actually on the same chromosomes (Table 2). Since the publication of Mendel's findings, other scientists have performed the pea plant crosses that could have shown linkage: i-a, v-fa, v-le, and fa-le. However, all of the pairs, except v-le, are so distantly located that Mendel would have been unable to detect linkage. Therefore, they behave as though they independently assort. The v-le cross, on the other hand, would have shown linkage if Mendel had completed the cross. Possibly, with just one more cross, Mendel would have discovered linkage.

Relationship between modern genetic terminology and character pairs used by Mendel

Character pair used by Mendel	Alleles in modern terminology	Located in chromosome
Seed colour, yellow–green	*I–i*	1
Seed coat and flowers, coloured–white	*A–a*	1
Mature pods, smooth expanded–wrinkled indented	*V–v*	4
Inflorescences, from leaf axils–umbellate in top of plant	*Fa–fa*	4
Plant height, >1m–around 0.5 m	*Le–le*	4
Unripe pods, green–yellow	*Gp–gp*	5
Mature seeds, smooth–wrinkled	*R–r*	7

Table 2: Genetic crosses performed by Mendel in modern terminology. The seven traits studied by Mendel are listed in column 1with a description of the possible phenotypes associated with each. Column 2 lists the letters used in modern genetics terminology to symbolize the alleles for each phenotype. Column 3 lists the chromosome on which the gene for each trait is found.

COMPUTER SCIENCE

Write a Java method that calculates the number of times a string of length 2 is the same and at the same position in 2 given strings.

For example:
Same("xyz","xyz") --> 2
Same("aaqppmm", "aayppm") → 3
Same("123","198") --> 0

Take 2 strings from stdin and use your method inside the 'main' method to calculate the number of matches.
http://www.tutorialspoint.com/java/java_strings.htm
http://www.tutorialspoint.com/java/java_string_substring.htm

Solution

```
import java.util.*;

public class Same{

   public static void main(String[] args){

          Scanner in = new Scanner(System.in);
          System.out.println("Please enter the first string");
          String a = in.next();
          System.out.println("Please enter the second string");
          String b = in.next();
          System.out.println("These strings have " + same(a,b) + " two-charachter matches"
);

   }

   public static int same(String a, String b) {

          // Will keep counting until we reach the end of a shorter string.
          int len = Math.min(a.length(), b.length());
          int count = 0;

          // Compare all substrings of length 2
```

```
        for (int i=0; i<len-1; i++) {
          String aSub = a.substring(i, i+2);
          String bSub = b.substring(i, i+2);
          if (aSub.equals(bSub)) {  // Use .equals() to compare strings
                count++;
          }
        }
        return count;
  }
}
```