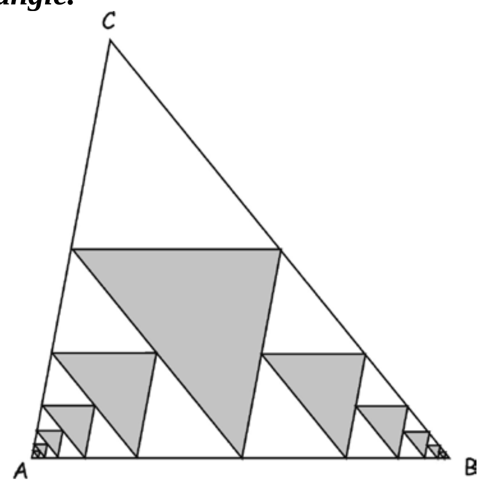
Problem of the Month.
Solutions: January 2013

MATHEMATICS

What fraction of the area of the triangle ABC does the infinite sequence of shaded median triangles occupy (see figure)?
A median triangle is obtained by connecting the midpoints of the sides of a given triangle.



Solution

Consider just one of the two sequences of shaded triangles - say, that extending right-down. For any triangle, the area of the median triangle equals ¼ of its area. The total shaded area of the sequence of shaded triangles is ⅓ of the area of the triangle ABC. Interestingly, this result can be obtained via several different arguments.
(i) Self-similarity (scale invariance). If the shaded area makes fraction f of the total area S of the triangle ABC, then $f = \frac{1}{4} + \frac{1}{4} * f$, so $f = \frac{1}{3}$.
(ii) Sum of infinite geometric series, $f = \frac{1}{4} + \frac{1}{4} * \frac{1}{4} + \frac{1}{4} * \frac{1}{4} * \frac{1}{4} + \ldots = \frac{1}{4} / (1 - \frac{1}{4}) = \frac{1}{3}$.
(iii) Simply looking at each row, one sees that it contains one shaded triangle and two un-shaded. Hence, ⅓ of all area is shaded.

Now, we add the area of the two sequences of shaded triangles shown in the figure, to obtain, $\frac{1}{3} + \frac{1}{3} - \frac{1}{4} = \frac{1}{3} + \frac{1}{3} * \frac{1}{4} = 5/12$.

PHYSICS

Two identical railroad cars are moving side-by-side on parallel horizontal tracks with the same initial velocity and without friction. Snow is falling (vertically). A person, standing on the roof of the first car facing forward, keeps brushing some snow off the roof by repeatedly moving a broom left-to-right and right-to-left (i.e. the snow leaves the roof perpendicular to the tracks as seen by the person). A person on top of the other car does nothing. Will either of the two cars get ahead, and, if so, which one? Explain your answer.

Solution

The car with a person brushing will fall behind. This can be most easily understood from the law of momentum conservation.

The snow falling on the roof of either car has to be accelerated to move with the car, thus it must acquire the momentum in the direction of motion. In the absence of external forces acting on the system, the total momentum of the car and the snow that fell on it is conserved, so the snow that gets stuck to the roof obtains its momentum from the car, decreasing the car's velocity.

More specifically, consider a snowflake of mass m that gets stuck to the car (perhaps with some snow already on the roof) of mass M. Conservation of momentum implies that

$MV0 = (M + m)V1$,

where $V0$ and $V1$ are the velocities of the car before and after the snowflake gets stuck to it. This gives $V1= MV0/(M + m)< V0$, i.e. the velocity of the car decreases. Furthermore, the relative change in velocity,

$(V1- V0)/ V0 =V1/ V0 -1= -m/(M + m)$,

is bigger (by absolute value) for the car whose mass M is smaller. Simply speaking, it is easier to decelerate a lighter car as it has less inertia. Since the car with the snow being brushed off is always lighter than the other car (i.e. its M is smaller), it decelerates faster and thus it falls behind. Of course, in reality one has to consider the cumulative effect of many snowflakes, however, as the above consideration applies to each, the conclusion remains the same.

What force is stopping the cars? It is the friction force between the snow (as it gets stuck to the roof) and the roof. This force acts on both cars, but, as explained above, it has a bigger effect on the lighter car (the one with the person brushing), since that car has less inertia.

Is there any other effect from snow-brushing, apart from making the car lighter? No, there is none, because the snow is swept off transversely to the car's motion, so the process of brushing includes no forces in the direction of motion. (Each chunk of snow brushed off the roof simply keeps moving in the forward direction with the same velocity as the car, plus with some transverse velocity component supplied by the brushing person.) The answer to the problem could, of course, change, if the snow was brushed off backwards, or, at least, with some velocity component directed backwards.

Finally, another elegant way to look at this problem is as follows (assuming, for simplicity, that the person manages to brush off all the snow off the car).

If the goal is to get the car running ahead, than the person brushing the snow off the roof is "wasting" the momentum acquired by that snow: each new portion of the snow gets its momentum from the car alone, while in the case where the snow accumulates on the roof, the momentum obtained by that snow from the car is "recycled" to share the momentum to the next portion of the snow. Hence, the car that accumulates the snow has less drain on its momentum, and therefore less deceleration.

CHEMISTRY

Suppose you invite a friend to your house. As you are having a snack in the kitchen, she asks you to demonstrate what a chemical reaction looks like. You immediately decide to impress her with your knowledge of chemistry, and you tell her: "Oh, I can show you quite a few reactions just with the substances found in my kitchen, laundry and living room!" What are the reactions you would demonstrate to her?

For each reaction, please:

- Name the substances could you use (for example, sugar, table salt, etc.).
- Write the chemical equations of the reaction.
- Describe the experimental conditions (for example, "I'll dissolve a teaspoon of X in a cup of water, add Y gradually and heat the solution").
- Describe the results you expect to obtain (for example "I expect a colour change from blue to white").

Solution

Different answers are possible, depending on which substances can be found in your home. Below are some of the reactions you can observe using the substances everyone can find in your kitchen/laundry/living room:

1. Interaction between baking soda (sodium bicarbonate) and vinegar (dilute acetic acid). This is a two step reaction. The first step is an exchange between the baking soda (a salt) and acetic acid (an acid). Acetic acid is stronger than carbonic acid, so it displaces carbonic acid from its salt:

$NaHCO_3 + CH_3COOH = H_2CO_3 + CH_3COONa$

As a result, a new acid (carbonic acid) and a new salt (sodium acetate) are formed. However, you cannot see this reaction. What you observe is a second step, decomposition of the carbonic acid:

$H_2CO_3 = CO_2 + H_2O$

Carbonic acid is unstable and it decomposes quickly onto water and carbon dioxide (a gas). As a result, you may observe either bubble formation or violent eruption (depending on the amount of the chemicals you will take, and on the rate of addition). If you want to do this experiment, we recommend to wear goggles, because, although these chemicals are relatively harmless, eye protection is still desirable.

2. Reaction between citric acid and baking soda. This reaction is similar to the previous one. Citric acid ($C_5H_8O_7$) is also stronger than carbonic acid, so it also will displace it from sodium bicarbonate. The result will be the same: evolution of carbon dioxide, which you can easily see.

3. Decomposition of baking soda in hot water.
Baking soda (sodium *bi*-carbonate) contains one carbonic acid residue per on atom of sodium. However, another salt exists, sodium carbonate (without "bi"), which contains *two* atoms of sodium per one carbonic acid residue (under "carbonic acid residue" chemists mean the molecule of an acid which lacks hydrogens). Since sodium bicarbonate is less stable than sodium carbonate, it decomposes upon heating to give sodium carbonate and one molecule of carbon dioxide.

$2\ NaHCO_3 = Na_2CO_3 + CO_2$

You can observe this reaction is you add baking soda to hot (>70°C) water, or by heating baking soda in an oven. Sodium carbonate is a relatively strong base, and you can use it for some interesting reactions.

4. Electrolysis of the solution of table salt (sodium chloride) using AA battery.
When you dissolve a table salt, sodium chloride (NaCl) in water, water molecules pull sodium and chloride ions apart, so the ions in solution move independently on each other. No NaCl molecules exist in water solution, and that is a common property of all (or, strictly speaking, *almost* all) salts. Since Na^+ and Cl^- are independent, we can separate them by applying electric field. You can easily do that if you connect two wires to "-" and "+" ends of standard AA battery and immerse other ends of each wire into the glass with a salt solution. As a result, Na^+ and Cl^- will start to move to negative and positive wire, accordingly. That is not a chemical reaction yes. The chemical reactions start when the ions met the wire's surface. The Na^+ ion will get an additional electron (e^-) from the wire and become metallic sodium

$Na^+ + e^- = Na$

However, sodium is very active, so it immediately reacts with water to form sodium hydroxide and hydrogen:

$2Na + H_2O = 2\ NaOH + H_2$

In other words, near the end of the “-“ wire you get hydrogen (gas) and sodium hydroxide. **Note, this is an industrial way to produce these two important chemicals.**

On the surface of the positive “+” wire you get more complex reaction. Firstly, Cl^- ion loses one its electron and becomes a neutral chlorine atom

$Cl^- - e^- = Cl$

Like sodium, chlorine atoms are *extremely* active, so they immediately react with water, or with the wire’s material, or with each other. When chlorine atom meets the surface of copper wire, it oxidizes it to form a dark film (it has rather complex composition, so we will not discuss it here). When chlorine atom meets another chlorine atom, they form a molecule of chlorine:

$Cl + Cl = Cl_2$

You can see evolution of this is a gas at the end of “+” wire, and you can feel it because it smells like a bleach.
If chlorine atoms meet water molecule they form hydrogen chloride and hydrogen hypochlorite:

H2O + Cl + Cl = HCl + HOCl

Both these substances are acids, so they react with sodium hydroxide (formed at the “-“ wire) to give a mixture of two salts:

$HCl + HOCl + 2\ NaON = NaCl + NaOCl + H_2O$

The solution of NaOCl is a bleach, and you can check it by adding some natural dye (from tea, red vine etc).

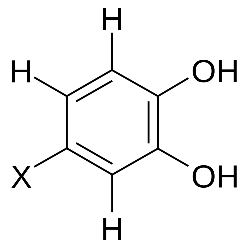
5. Decomposition of sugar.
Sugar is a carbohydrate, which means that it contains only carbon, hydrogen and water atoms, and the ratio between them is 1 : 2 : 1. One molecule of sugar decomposes onto carbon and water upon heating:

$C_{12}H_{24}O_{12} = 12\ C + 12\ H_2O$
You can easily do that reaction by heating sugar in a metallic spoon. However, we do not recommend you not to do that, because when heating is not strong enough, the products of partial decomposition of sugar may form, which have a very unpleasant odour. Although that is absolutely safe, your parents will not be pleased.

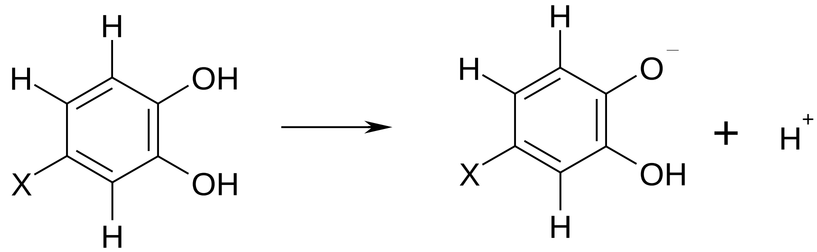
6. Burning of methane.
Of course, if you have a gas stove in your house it is easy to demonstrate this reaction:

$CH_4 + 3\ O_2 = CO_2 + 2\ H_2O$

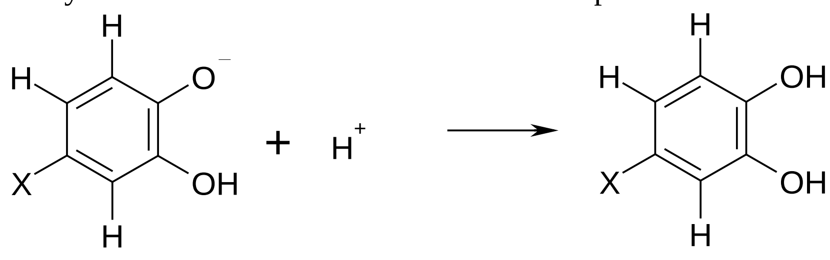
7. Change of color of black tea (from brown to yellow) after addition of lemon (or citric acid); change of color from brown to deep brown after addition of baking soda.
Green and black tea contains several chemical compounds that are responsible for its antioxidant properties. These compounds are called catechins, and they have this general formula:



X denotes a part of the molecule that is not important for us. For simplicity, carbon atoms are not shown, so each vertex of the hexagon actually is an atom of carbon (that is how the chemists usually draw formulas of organic compounds).
The catechins are weak acids, so in their water solution (for example, in tea) they fall apart:



For us, this reaction is interesting because the starting material has just a slightly yellow colour, but the resulting negative ion is brown, so when this reaction occurs it can be easily monitored. The reverse reaction is also possible:



In that case the colour changes from brown to yellow.
How can we do that reaction? Prepare three cups of black tea. To one cup add a slice of lemon (which contains citric acid), to another cup add a tea spun baking soda (which is a base), and the third cup leave as a control (for comparison). Look at the change of colour. Depending on the concrete brand of tea you will be able to see a change of colour from brown to deep brown upon addition of soda, and you always will see the colour change

when you add lemon. Interestingly, this reaction is reversible, so if to the first cup you add large amount of soda, the colour will change back from yellow to brown.

8. Preparation of soap.

Fats (butter, oil, etc) can be hydrolyzed when heated with water solution of strong bases, and the product of this reaction is an ordinary soap. Since you already prepared sodium carbonate (which is a rather strong base), you may try to do this reaction. You may take a solution of sodium carbonate prepared according to the reaction #3 (this solution must be concentrated), add small piece of butter, and gently boil it until butter completely dissolves. Since butter is not soluble in water, the fact that butter dissolved is an indication that some reaction occurred. Let this solution to cool to the room temperature and examine it. The solution will look like a soap solution, and that is not a coincidence: that is really soap.

Of course, this list is not exhaustive. You may propose other reactions, and if you will be able to do that you will get extra points.

BIOLOGY

It is known that milk of mammals varies in its fat content from as low as 2% in, for example, monkeys, wolves, and cows to as high as 65% in whales, polar bear, and hares. How would you explain this fact? Provide at least 2 additional examples of animals having low and high milk fat content and explain why do you think it is happening in each particular case.

Solution

Milk is the only natural complete food which exist and each milk is adapted to assist the growth of its particular species.

Each species of mammal makes a unique kind of milk, which meets all the nutritional requirements of its offspring at the beginning of life. Each species' milk has specific qualities that insure the survival of the young in a particular environment. This principle is known as the biological specificity of milk.

Some animals (hares, seals) must leave their offsprings for a long period of time and cannot feed them frequently. Their milk must be more nutritious and higher in the fat content compared to others.

Mammals living in the water (whales, seals, otters, dolphins, etc. - a diverse group of 128 species) must have a higher fat content in the milk, because their bodies lose heat faster than the same sized land animals. Therefore animals living in the water usually create a thick layer of fat. Hooded seals are most impressive in this regard with a 4 days lactation intervals and a milk fat of 65%.

Marine mammal youngsters are very active since they must follow their parents from the first day of their life. The rapid growth and active lifestyle of marine mammal offsprings is made possible by the ingestions of extremely fat-rich milk.

With a few noticeable exceptions, land (terrestrial) animals produce milk that is

low in fat. The milk of a polar bear typically has a higher fat content than that of other bears. Polar bear's milk is more similar to that of marine mammals than it is to milk of most terrestrial mammals (see above).
Offsprings of most of the land animals usually live together with their mother who can provide milk at any time. This is one of the reasons why they don't need to have a fat milk.
Most animals living in hot climate have a low fat milk content (camel, kangaroo, some goats).
The fact that human milk contain only about 4 % of fat has by some been regarded as evidence of a tropical origin of human race.

Examples: see table below.

Species:	Water, %	Fat, %	Casein, %	Whey protein, %	Lactose, %	Energy, kCal/100g
Black Bear	55.5	24.5	8.8	5.7	0.4	280
Camel	86.5	4.0	2.7	0.9	5.0	70
Cow	87.3	3.9	2.6	0.6	4.6	66
Dog	76.4	10.7	5.1	2.3	3.3	139
Dolphin	58.3	33.0	3.9	2.9	1.1	329
Donkey	88.3	1.4	1.0	1.0	7.4	44
Elephant	78.1	11.6	1.9	3.0	4.7	143
Fur Seal	34.6	53.3	4.6	4.3	0.1	516
Goat	86.7	4.5	2.6	.6	4.3	70
Horse	88.8	1.9	1.3	1.2	6.2	52
Human	87.1	4.5	0.4	0.5	7.1	72
Kangaroo	80.0	3.4	2.3	2.3	6.7	76
Pig	81.2	6.8	2.8	2.0	5.5	102
Rabbit	67.2	15.3	9.3	4.6	2.1	202
Reindeer	66.7	18.0	8.6	1.5	2.8	214
Sheep	82.0	7.2	3.9	0.7	4.8	102
Water Buffalo	82.8	7.4	3.2	0.6	4.8	101