PROBLEM OF THE MONTH

November 2013

MATHEMATICS

Prove that $n^5 - 5\,n^3\; + 4n$ is divisible by 120.

Solution

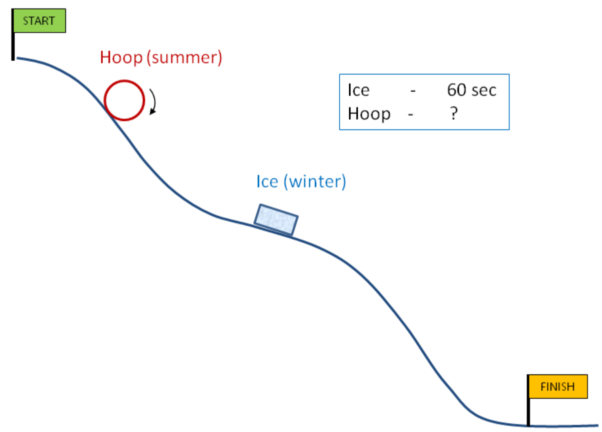
It is easy to find that

$$n^5 - 5\,n^3\; + 4\,n\; = \; n(n^4 - 5n^2 + 4) = n(n^2 - 1)(n^2 - 4) = (n-2)(n-1)n(n+1)(n+2)\; .$$

The given number is a product of five consecutive integer numbers. One of these numbers is divisible by 5 and one of them is divisible by 3. Also, these five numbers contain at least two consecutive even numbers. One of the latter is divisible by 4 and the product of these even numbers is divisible by $2 * 4 = 8$. As a result we obtain that the given number is always divisible by $5 * 3 * 2 * 4 = 120$.

PHYSICS

In winter, a flat piece of ice resting on the top of the mountain starts sliding down an icy track. The track has rather irregular shape, and its slope is not constant (see the picture). It takes t=60 seconds for the piece of ice to reach the bottom of the track.
In summer, an experiment is made with a hoop that rolls down the same track with zero initial speed without any sliding.



a) Will it take more, less or the same time for the hoop to complete the track? Substantiate your answer.

b) Can you find out, how much time (in seconds) it will take for the hoop to complete the track?

Note that kinetic energy of a rolling object is the sum of the kinetic energies of its translation and rotation. Neglect air resistance and all other losses of energy due to friction.

Solution

a) For simplicity, let us assume first that the ice block and the hoop both have the same mass **m**. The energy is conserved, so in both cases potential energy completely converts to kinetic one. Therefore, when the block and the hoop pass the same point they have **the same kinetic energy**. However, in the case of the hoop, this energy is distributed between rotation and translation, as opposed to the case of ice block, which only has translational motion without rotation. This means that the block moves faster at any point of the track, than the hoop when it passes the same point. Hence, the total time of travel for the rolling hoop must be larger.

b) We need to find how the kinetic energy of a hoop is distributed between the translation and rotation. If the hoop moves with speed **v,** its kinetic energy due to the translational motion has familiar form:

$$K_{tran} = \frac{mv^2}{2}$$

At the same time, each point of the hoop rotates about its center. All the points of the hoop are moving with the same speed **v** in the reference frame of the center. Therefore, rotational kinetic energy is exactly the same as translational one (this is only true for a hoop, and would be different for a disk or a ball!):

$$K_{rot} = \frac{mv^2}{2}$$

Therefore, the total kinetic energy of the hoop is

$$K_{hoop} = K_{tran} + K_{rot} = mv^2$$

On the other hand, the ice block moving with speed v only has translational kinetic energy:

$$K_{ice} = \frac{mv^2}{2}$$

When the hoop and the block pas the same point of a track, they have the same kinetic energy:

$$K_{ice} = K_{hoop}$$

Therefore,

$$mv_{hoop}^2 = \frac{mv_{ice}^2}{2}$$

We can see that at any point the hoop will move slower than the ice block by factor 2:

$$v_{hoop} = \frac{v_{ice}}{\sqrt{2}}$$

Naturally, the total travel time will be increased by the same factor:

$$t = \sqrt{2} \cdot 60\ sec \approx 85\ sec$$

To refine the solution, we have to recall that we assumed the masses of the block and the hoop to be the same. This is not an important assumption. Both potential energy in gravity and kinetic energy are proportional to mass, so the mass always cancels. Let **H** be the height of the mountain, and **h** be the height at certain point of the track. The corresponding potential energy difference for an object of mass **m** is $mg(H-h)$. As we already discussed, this potential energy is converted to kinetic. So, in the case of the ice block of mass m_{ice}, we obtain:

$$m_{ice}g(H-h) = \frac{m_{ice}v_{ice}^2}{2}$$

As expected, m_{ice} will cancel. Similarly, the mass of the hoop will cancel for the corresponding energy balance equation:

$$m_{hoop}g(H-h) = m_{hoop}v_{hoop}^2$$

From these two equations, we can once again obtain the earlier result: $v_{hoop}b = v_{ice}/\sqrt{2}$, this time without assuming the same mass of the hoop and the ice block.

CHEMISTRY

On July 2, 1982, Larry Walters, a truck driver from Southern California, made his only daredevil historic solo flight. He fastened 42 helium filled weather balloons to a lawn chair and anchored it with a rope to a bumper of his jeep. He carried various supplies with him as well as a CB radio and a BB gun to shoot balloons one at a time to descend. Unfortunately, he dramatically underestimated the buoyancy of the balloons: when he cut the rope, he took off so quickly that in few minutes he reached altitudes of up to 16,000 feet, and for several hours he was floating in skies above L.A. According to an article in the New York Times, Walters was spotted by pilots from TWA and Delta Airlines, who radioed the tower that they passed a guy in a lawn chair with a gun! It was cold at 16,000 feet and he started shooting some of his balloons to descend, but dropped his BB gun and had to wait for his rig to come down on its own.

Suppose (theoretically) you are going to reproduce that flight, and you obtained needed amount of empty balloons, but you have no helium to fill them (your parents didn't approve your enterprise, and they did not allow you to buy helium). Therefore, you decided to use hydrogen instead of helium. You have an unlimited amount of battery acid and common materials. How many pounds of iron nails do you need to buy at a hardware store to take off sitting in a chair (assuming that chair's weight is 2 kg, and other parts of your "apparatus" have zero weight)?

To solve this problem, you may use information from the October PoM, and from the SchoolNova web site (http://schoolnova.org/nova/classinfo?class_id=chemistry101&sem_id=f2013), or to look up molecular masses and/or densities of various substances in the web.

Solution

To produce hydrogen needed to lift you to the sky, you need iron (iron nails) and an acid. The battery acid (sulfuric acid, H_2SO_4) is arguably the only strong acid you can find in a store. Therefore, the equation of this reaction is as follows.

$$Fe + H_2SO_4 = FeSO_4 + H_2 \qquad (1)$$

Of course, other reactions exist that may allow you to produce hydrogen at home. For example, you can produce hydrogen by electrolysis of water, or via decomposition of hydrogen peroxide, etc. However, all those methods will hardly allow you to prepare needed amount of hydrogen, and to inflate weather balloons.

Let's assume we collected all needed stuff in your garage: bottles with battery acid, a lawn chair, ropes, and weather balloons. We need to calculate the amount of iron to produce hydrogen. There are two approaches to calculate that amount, a lazy man's approach, and a brute force approach. Being lazy, I prefer the first method.

As we know, 2 grams of gaseous hydrogen occupy 22.4 L volume at standard pressure and temperature[1]. That means 2 grams of hydrogen displace 22.4 L of air, whose "molecular mass" is

[1] We can assume this volume is about 22.7 L at summer.

approximately 28.8[2]. In other words, a balloon filled with 2 grams of hydrogen is capable of lifting up to 28.8 - 2 = 26.8 gram weight (29 g is a mass of a displaced air, and 2 g is a mass of hydrogen). Using the equation (1), we can calculate the amount of iron needed to lift a 26.8 gram weight: since one iron atom (with atomic weight of 56 Da) produces one molecule of hydrogen (with molecular weight of 2 Da), 56 grams of iron produce 2 grams of hydrogen. In other words, 56 grams of iron, upon dissolution in a battery acid, produce the amount of hydrogen sufficient to lift a 26.8 gram weight.

The problem is almost solved. Just the last step is remaining. Let's assume your weight is 48 kg. Together with a chair, it will be 50 kg. To calculate the amount of nails, set a simple proportion:

56 g are needed to lift a 26.8 g load
x kg are needed to lift a 50 kg load

Obviously, $x = 50(56/26.8) \approx 104\,kg$, or approximately 230 lbs. Of course, if you really want to reproduce such a flight, you need to buy at least 250 lbs to compensate for possible hydrogen leaks and seasonal variations of air density. However, to do that would be a very bad idea, so we strongly recommend you to refrain from that ☺.

The second solution is much longer. Firstly, you need to find in literature the density of air (about 1.21 g/L; it is senseless to look for a more precise number, because air density depends on temperature, humidity, etc.) Then, find a density of hydrogen (approximately 0.089 g\L; again it depends on the atmospheric pressure and temperature). Now we need to calculate Archimedes' force acting upon a 1 L balloon filled with hydrogen. It is equal to the weight of 1 L of air minus the weight of 1 L of hydrogen: a 1 L balloon filled with hydrogen can lift a 1.21 - 0.089 ≈ 1.2 gram weight. Now, when we established the buoyancy of 0.089 grams of hydrogen, we can calculate the amount of hydrogen needed to lift a 48 kg PoM participant along with his lawn chair (50 kg). To do that, let's set a proportion:

0.089 grams of hydrogen lift a 1.2 g weight
x grams of hydrogen lift 50 kg
x = 50*0.089/1.2 = 3.71 kg.

To calculate the mass of iron needed to produce 3.71 kg of hydrogen, let's use the equation (1), and solve a second proportion:

56 g of iron are needed to produce 2 g of hydrogen
y kg of iron are needed to produce 3.71 kg of hydrogen
y= 56*3.71/2 ≈ 104 kg.

Note, both methods gave approximately the same result, but the first method is much more simpler. What conclusion can be drawn from that?

BE LAZY GUYS! Try to think instead of doing unnecessary work.

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[2] Air is a mixture of gases, not an individual substance, so it would be not completely correct to speak about its molecular mass. However, we can speak about some "effective" molecular mass. Air is composed primarily of oxygen (ca 20%) and nitrogen (ca 80%), so its effective molecular mass is 0.2*32+0.8*28=28.8.

BIOLOGY

Imagine yourself a biologist on the board of Tara*, a unique, specially equipped, 36-meter-long sailboat designed for the most advanced scientific experiments, in the middle of Indian Ocean. You study green microscopic algae. In the morning, you took a metal bucket and scooped a full bucket of ocean water and placed a bucket on the deck of the boat. You measured the amount of green microscopic algae in the bucket immediately and in the evening and noticed that the amount of green microscopic algae has increased significantly. Why do you think this has happened? Please, name as many factors that may influence the algae growth as possible. Try to select the most important factors (say 3) and propose experiments to test each of them.

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*"Our seas and oceans are under constant threat. They are affected by many kinds of human activity, by coastal erosion, CO_2 and acidification, transport activities and climate change. But they are also an enormous source of wealth, much of which is still undiscovered". (European Commission's Research Directorate-General). To investigate all these issues, an international team of oceanographers, ecologists, biologists and physicists from prestigious laboratories set out on a three-year, round-the-world voyage (known as Tara Oceans Expedition), with the goal of analyzing plankton ecosystems in relation to physicochemical conditions throughout the world's oceans, assessing their adaptation to a rapidly changing earth system. For more information check out the Tara Oceans Expedition website at http://oceans.taraexpeditions.org/

Solution

As a biologist you know that green algae are photosynthetic organisms that can be found in many habitats ranging from fresh water to salt water and from the poles to the equator. They are simple organisms that range in size from the one-celled microscopic types to large seaweeds that grow to over 230 feet. They are also hardy organisms and have a tremendous reproductive capacity. They have adapted to all kinds of water conditions and provide oxygen and food for aquatic life. They can be *planktonic*, meaning they float freely in the ocean and when they do, they are called *microalgae*. The term *seaweed* refers to larger species of so-called *macroalgae*, which live in the marine environment attached to the bottom.

Green algae are photosynthetic so they use carbon dioxide and convert it into oxygen.
Algae need light to photosynthesize and they produce carbon dioxide during respiration.
Check out some of the cool images of micro- and macroalgae at http://www.ohio.edu/plantbio/vislab/algaeimage/Chlorophyta.htm
http://www.telegraph.co.uk/science/space/9125409/The-algae-bloom-so-big-it-can-be-seen-from-space.html

For all their diversity, they have one thing in common: light, temperature and nutrients are the key components for algae intensive growth. Such critical components are called limiting factors because they limit the growth or development of an organism or a population. In biology, factors mentioned above are called **abiotic** - non-living chemical and physical factors in the environment, which affect ecosystems.

So, ***the most important ABIOTIC factors*** that can affect algae growth in the bucket of water you have scooped are:

Light. Green algae need light to grow just like your favorite flower or backyard tree does. Algae have a better access to sunlight in the bucket on the deck because bucket is shallower and algae receive constant amount of light (not like in the ocean where they float between different layers of water due to the currents).

Temperature. The temperature of the water in the bucket will be higher compared to the ocean – the small volume is getting warmer faster and there are no ocean currents in the bucket. Higher temperature facilitates cell growth. The optimal temperature for phytoplankton cultures is generally between 20°C and 24°C. Most commonly cultured species of micro-algae tolerate temperatures between 16°C and 27°C. Temperatures lower than 16°C will slow down growth, whereas those higher than 35°C are lethal for a number of species.

Nutrients. Keep in mind that photosynthetic organisms are still living things, with protein-based chemistry, which means that they have nutritional requirements beyond carbon dioxide and water. Proteins contain a significant amount of ***nitrogen***, which usually needs to be absorbed as ***nitrates*** (a nitrogen-oxygen molecule) to be usable. Plants convert the nitrates into ***amino acids,*** which are the components of ***protein molecules.*** The production and use of glucose for energy also requires ATP as an energy carrier; ATP contains ***phosphorus,*** usually absorbed as ***phosphates*** (a phosphorus-oxygen molecule). Anyone who takes care of plants knows that nitrates and phosphates are important ingredients in fertilizers. Most photosynthesizers have other nutrient needs: they make a few critical molecules with materials such as iron, or need small ions, such as sodium, for some of their chemical processes). As a matter of fact, an excess of iron in the water around Antarctica is considered one of the critical contributors to algae blooming in that area: http://www.timeslive.co.za/scitech/2012/03/05/monster-algal-bloom-spotted-in-antarctica.

Suggested experiments:

When we study the influence of one factor, we should create conditions under which the rest of the factors remain the same. When we check the **influence of light** on the growth we can't just take a second bucket and keep it in the dark – we have to keep the temperature the same. We either should cover the 2nd bucket or keep the first one in the shadow – no direct sunlight that will increase the temperature.

To check the **influence of temperature** we have to keep a second bucket at lower temperature. We can place our bucket inside another bigger bucket with cold water and try to maintain the constant temperature similar to the ocean water temperature for the entire day and compare cell growth.

For the **role of nutrients** we can add extra nutrients in the water of the bucket and compare cell growth. Organisms usually need a very small amount of nutrients; therefore we should add just very tiny amounts.
What are other abiotic factors that can affect an intensive algae growth?

Shaking. This may also play an important role in cell growth, because it gives cells an access to more nutrient resources, thus speeding up a number of metabolic processes. From other side, high shaking may destroy the algae.

Surface attachment. Many algae species can grow better if attached to the solid surface. (However, it is rather unlikely to collect those "attaching" algae in the middle of the Indian Ocean).
PH Level. Higher pH level (alkaline conditions) creates a perfect environment for algae to grow in. The pH range for most cultured algal species is between 7 and 9, with the optimum range being 8.2 - 8.7. Values of over pH 9 can limit the growth. This could be the case in our bucket when population lacking additional aeration becomes dense enough. So high pH is a limiting factor but most probably it will not play any role in the interval of just one day.

Algae are not the only living things in the ecosystem, therefore we have to consider the **BIOTIC factors** – all other living things that shape an ecosystem.

Algae Eaters. Algae are the food source for many animals forming zooplankton. Tangs, angelfishes, some blennies, rabbit fishes, sea urchins, hermit crabs, and many kinds of snails are in this category. If the balance between phyto- and zoo-plankton is shifted towards the algae (or the plankton animals are more sensitive to the changed environmental conditions), microscopic phytoplankton will benefit.

Microscopic vs. macroscopic. Small cells divide faster and will outgrow macroscopic species. Algae population is heterogeneous and because of the faster metabolism, the microalgae will grow faster than macroalgae.
And it is also possible that algae will grow with the same rate both in ocean and in the bucket on the deck of the boat. It could happen because you have scooped your sample in the morning when algae growth is minimal (no light and cooler temperature during nighttime).

There are many factors that may influence algae growth. In reality, not a single factor but a combination of them will affect how fast or slow these organisms grow.

COMPUTER SCIENCE

1) 5 points

Write a Java program to compute the following

1/100 + 2/99 + 3/98 + … + 99/2 + 100/1

using a loop. Print out the result and try to make your program as short as possible.

2) 5 points

Write a Java program that defines and then uses a method (function) to return the volume of a sphere given its radius.
Use java.lang.Math for the value of PI.
Call your method from the main method 3 times with different arguments and print out the results.

Resources:

http://csc.columbusstate.edu/woolbright/java/loops.html
http://www.tutorialspoint.com/java/java_loop_control.htm
http://math.hws.edu/javanotes/c2/s5.html
http://www.tutorialspoint.com/java/java_methods.htm
http://www.horsesnw.com/articles/JavaMethod.htm

Solution

```
public class POM_November_CS {
      public static double SphereVolume(double radius){
          return 4.0/3.0 * Math.PI * Math.pow(radius,3);
  }
      public static void main(String[] args)  {
          float sum = 0;
          for(float i = 1,j=100;i<101;i++,j--){
             sum += i/j;
          }
    System.out.println("Sum of 1/100 + 2/99 + ... + 100/1 =  " + sum );
    System.out.println("Volume of a sphere with the radius 2.6 = " + SphereVolume(2.6));
    System.out.println("Volume of a sphere with the radius 5 = " + SphereVolume(5));
    System.out.println("Volume of a sphere with the radius 7 = " + SphereVolume(7));
  }
}
```