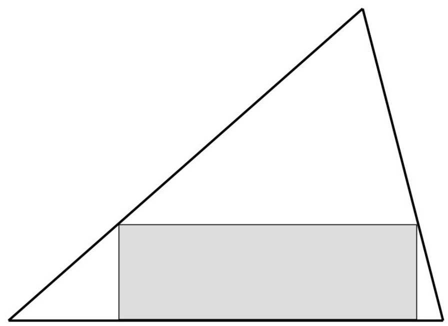
PROBLEM OF THE MONTH

October 2013

2013-2014

MATHEMATICS

A rectangle is called inscribed in a given triangle when all four vertices of the rectangle belong to the sides of the triangle (see an example in the figure)



a) Given a triangle construct the rectangle inscribed into it, which has the largest area.

b) If the area of a given triangle is S what is the area of the maximal rectangle inscribed into it?

Solution

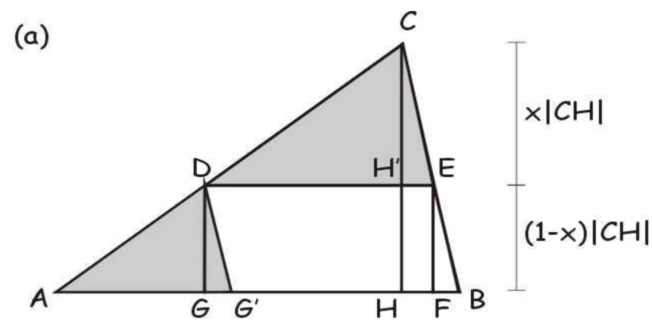
Figure below illustrates two different solutions to this problem:

- one using some algebra, (a)
- and the other purely geometric, (b), (c).

In both cases it is convenient to consider the total area of the remaining triangles and find when it is minimal.

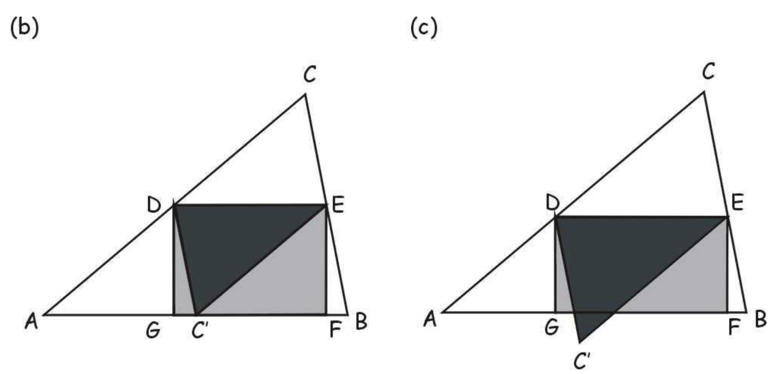
(a) Noting the following similar triangles, $\Delta DEC \sim \Delta ABC \sim \Delta AG'D$, where $\Delta AG'D$ is a union of ΔAGD and $\Delta GG'D \simeq \Delta FEB$, we see that the total area of the three remaining triangles is equal to the area of the two shaded triangles, ΔDEC and $\Delta AG'D$, which are both similar to ΔABC. The similarity coefficients are $|H'C|/|HC| = x$, and $|H'H|/|HC| = 1 - x$, respectively. Thus, the remaining area is $x^2 + (1-x)^2$ times the area S of ΔABC. This expression is smallest for $x = 1/2$, which can be seen by applying the inequality between the quadratic mean (root-mean-square) and the arithmetic mean, $\left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}} > \frac{a+b}{2}$, or directly, by isolating the full square as shown in the Figure (a). Hence, the smallest area of the remaining triangles and, therefore, the largest possible area of the inscribed rectangle, equals half the area S of ΔABC.

(b), (c) By drawing lines $DC' \parallel BC$ and $EC' \parallel AC$ we obtain three triangles (shaded in the Figure), which are congruent to the three remaining triangles: $\Delta\mathrm{DEC} \simeq \Delta\mathrm{DEC}'$, etc. When DE is the median, all three shaded triangles fit exactly into the rectangle $DEFG$, so their area is equal to that of the rectangle and is half the area S of $\Delta\mathrm{ABC}$. Otherwise, either the $\Delta\mathrm{DEC}'$, as in Fig. (c), or the other two triangles (make a drawing yourself), extend outside the rectangle, so that the area of the rectangle is smaller than that of the remaining triangles, and is less than half of the area S of $\Delta\mathrm{ABC}$.



$S_{AGD}+S_{EFB}+S_{DEC} = S_{AG'D}+S_{DEC} = x^2 S_{ABC}+(1-x)^2 S_{ABC} >= \frac{1}{2} S_{ABC}$

$x^2+(1-x)^2 = 1-x+2x^2 = \frac{1}{2}+2(x-\frac{1}{2})^2 >= \frac{1}{2}$



DC' || CB, EC' || AC, $S_{DEC} = S_{DEC'}$

$S_{AGD}+S_{EFB}+S_{DEC}$ = sum of the areas of shaded triangles >= $\frac{1}{2} S_{ABC}$

PHYSICS

A bullet of mass *m* hits a back of a swivel chair of mass *M*>>*m* and gets stuck inside. In one case the bullet hits a point located on the centerline (point A), while in the other case it hits closer to one side (point B).

Qualitatively describe the resulting chair motion in both cases. In which case it will travel faster with respect to the floor? Where does the momentum of the bullet go? Where does its energy go?

Assume that the back of the chair can freely rotate with respect to the base, which in turn can freely roll along the floor. The bullet can be assumed travelling horizontally, and hitting the chair head-on. Point A can be assumed to be the center of mass of the chair.

Solution

When the bullet hits the center of mass (point A), the chair ends up travelling as a whole along a straight line given by the bullet direction. In case B there is, in addition, some rotation of the back of the chair with respect to the base. However, in both cases, the chair's final center of mass velocity is exactly the same. This directly follows from the conservation of momentum. As usual with any light object inelastically colliding with a heavy object, most the momentum goes into the heavy object, while most of the energy gets dissipated into heat.

More quantitatively, from the momentum conservation, denoting the initial bullet velocity by V_0, and the center of mass velocity after the bullet is stuck by V_1, we get

$$mV_0 = (m + M)V_1 \approx MV_1,$$

where the right hand side follows from the $M \gg m$ condition. Note that this equation and thus the final (center of mass) velocity V_1 is exactly the same in both cases A and B. Also note that the momentum of the bullet essentially transfers into the momentum of the chair.

What about the energy conservation? Of course it applies here as well. Initially there is only the kinetic energy of the bullet. After the bullet is stuck there is 1) kinetic energy of the uniform motion; 2) (case B only) kinetic energy of the rotational motion, denoted below by K_{rot}; and 3) the heat (denoted by Q) that the bullet deposits into the chair material. The energy conservation law is then

$$\frac{1}{2}mV_0^2 = \frac{1}{2}(m+M)V_1^2 + Q_A \approx \frac{1}{2}MV_1^2 + Q_A\ , \qquad \text{(case A)},$$

$$\frac{1}{2}mV_0^2 = \frac{1}{2}(m+M)V_1^2 + K_{rot} + Q_B \approx \frac{1}{2}MV_1^2 + K_{rot} + Q_B\ , \quad \text{(case B)}.$$

Expressing the final velocity from the momentum conservation law above, $V_1 \approx V_0 m/M$, we immediately see that the final kinetic energy term is very small, as compared to the initial energy, $\frac{1}{2}\ MV_1^2 = \frac{1}{2}mV_0^2 m/M \ll \frac{1}{2}mV_0^2$. This means, that almost all of the initial energy of the bullet goes into heat, specifically, for case A, $Q_A\ \approx \frac{1}{2}mV_0^2$.

In case B, some of the initial energy also goes into K_{rot}, and therefore the generated heat is smaller than in case A. While there is not enough detail in the formulation of our problem to exactly relate K_{rot} to the initial energy (at a minimum one would need to know the mass distribution between the various parts of the chair), we can easily estimate that, for any realistic chair, K_{rot} is small, more specifically $K_{rot} \ll \frac{1}{2}mV_0^2$. This implies that, similarly to case A, almost all of the initial energy of the bullet goes into heat, i.e. $Q_B\ \approx \frac{1}{2}mV_0^2$.

Finally, we note that the fact that (very slightly) less heat is generated in case B could be understood even without applying the energy conservation. Indeed, if the chair is rotating as the bullet is penetrating through it, the (time-averaged) relative velocity of the bullet with respect to the chair material is smaller than in case A. This reduces the friction, and the heat generated from it.

CHEMISTRY

We know from experiments that 22.4 L of every gas at room temperature and atmospheric pressure contain 6×10^{23} molecules. We also know that baking soda, when mixed with acids, for example, with acetic acid, produces one molecule of carbon dioxide gas per one soda molecule according to the following equation:

$$NaHCO_3 + (some\ acid) = CO_2 + (some\ salt)$$

Can you propose an experiment that would allow you to calculate the number of sodium atoms in one teaspoon full of baking soda?

You can use any standard equipment you can find in your home: rulers, compass, toys, Coke bottles, citric acid, vinegar, lemon juice, balloons, latex gloves, electric tape, screw drivers, pliers, scotch tape, etc (of course, you don't need to use all of that). If you have electronic scale in your kitchen, you can take not a teaspoon full, but 10 grams of soda.
You may just describe the experiment, or perform it (of course, in that case, you will get more points). In the latter case, please send a photo of your experimental setup.

IMPORTANT! Although you will be dealing with ordinary substances, not with lab chemicals, please, wear goggles before starting your experiment.

Solution

Pour about 100 mL of vinegar into a 0.33 L Coca Cola bottle (It is important to take an excess of an acid; since vinegar is a 5-8% solution of acetic acid you need to use at least 100 mL to make sure all soda has reacted). Place pre-measured amount of soda into a balloon (or into a latex glove; in the latter case, your measurement will be less accurate), and attach the balloon to Coca bottle's neck tightly (for example, using a rubber ring or an electric tape). Tip the bottle, pour its content into a balloon, and shake well. When evolution of the gas ceased, measure balloon's volume. You can do that using three approaches.

(i) Measure balloon's circumference (L) using a measuring tape. Assuming the balloon's shape is spherical, its volume V is equal to $pi * d^3/6$, and the balloon's diameter (d) is equal to L/pi. Therefore, the volume of the balloon is $V = L^3/(6 * pi^2)$.
(ii) You can simply measure the balloon's diameter with a ruler, and use this the formula $V = pi * d^3/6$ directly, although such a measurement will be less precise.
(iii) You also can measure the volume of water displaced by the balloon (for example, by immersing the balloon into a basket with water, and by measuring the change of the water level). Now, after you determined the gas volume, let's start to calculate a number of atoms. We know that 22.4 L of carbon dioxide contain 6×10^{23} molecules, therefore, there will be $N = (V/22.4) \times 6 \times 10^{23}$ molecules of gas in the balloon. Since one molecule of the gas corresponds to one sodium atom in the soda we took, the amount of sodium atoms will be equal to the amount of gas molecules. In other words, a solution is $N = (V/22.4) \times 6 \times 10^{23}$.

BIOLOGY

The dawn chorus is one of the most conspicuous vocal behaviors of birds. Near sunrise, birds often sing more loudly and vigorously than they do at other times of the day.

Why do you think this is the case?
What are the potential benefits of such behavior?

Please provide as many possible explanations (reasons) as possible.

Solution

1. Why and when birds make sounds.
The purpose/reason of basically any kind of "sounds" are for contacts with outside world (social signaling):
   a) It could be the signal, saying "I am alive, survived the night" or "I am back from migration"
   b) "this is my territory" or "I am single, strong, healthy and ready for offsprings". (get-off or come on)
   c) the signals about danger, or the signal about a whereabouts of a bird or birds flock.
   d) some birds use song as a form of defense against predator birds. By singing, the skylark signals
to the merlin that it is in good condition and will be a hard job to catch. Only strong birds with extra energy and strength can invest the energy need for loud, continuous singing, and evade any predator that may detect it.

2. Why many species often do it on the dawn:
Dawn is the best time to sing because:
   a) less predators
   b) less food - Light is poor and insect prey is not flying, so foraging for food is difficult.
   c) the sound goes further (less background noise and qualities of the air): A dawn song is thought to be 20 times more effective than singing at midday

d) announcing that survived the night (if any birds have died overnight, the others will know where there is a vacant territory.)

e) female birds generally lay eggs in the morning, so a dawn mating is the best time for a male

3. Why sing in the spring (mostly):

- Breeding behavior is stimulated by a change of brain chemicals within the males, notably by a rise in testosterone and a fall in melatonin. Melatonin is regulated by daylight and its levels decrease when the days get longer. This reaches a point, which triggers breeding activity, and the dawn chorus, or battle of the songs, begins.

4. Why do many birds sing on the same time (chorus):

actually birds try to start singing as early as possible: The dawn chorus is thought to be a sign of male quality. High-ranking and older males usually start singing earlier in the morning and earlier starting males have more mating partners. It is observed that delaying or even skipping song at dawn has detrimental effects on male paternity.

Distinguish "bird song" from "bird call"

- A song is a stereotyped vocalization that is usually long and complex and is used to defend territory and/or attract a mate.
- Calls are basically all of the vocalizations that a bird makes other than its songs. Calls are usually short, simple vocalizations that are used in many contexts such as predator alarm, feeding, courtship, aggression, and social contact.

Calls are used mostly for the signals about danger, or the signal about a whereabouts of a bird or birds flock.

COMPUTER SCIENCE

Hi, if you never wrote a line of code in your life but is curious to see if it's fun to program, please go to the following web page that will let you write and execute Java programs right in your favorite Web browser:
http://ideone.com/

There you can put your code below the line 12 and click on the 'run' button to execute it.
For the first assignment you would learn about variables and how to display results to the screen. There are many tutorials on the Web that you could look at, but you also might start by copy-pasting the following example lines that should be pretty clear from the comments (the lines that start with a double slash: //) and just see how the program runs:

```
// Here is how we print in java:
System.out.println("It's fun to program! ");

// printing a variable of type String
String mys = "I actually like writing code...";
System.out.println(mys);

// Before we use a variable we need to declare it
int var1;  // var1 will hold an integer

// Let var1 be 10
var1 = 10;
// let's declare some more variables
int var2,sum;

var2 = 30;
sum = var1 + var2; // now sum should be 40, let's see.
The println method can only print strings,
// so we convert an int to a String
System.out.println(" sum = " + Integer.toString(sum));
```

Run the above code by clicking the 'run' button, try to modify it, play with it by add more variables of different types (read for example http://www.tutorialspoint.com/java/java_basic_datatypes.htm).

Then when you understand what's going on, add the following 4 lines to the code and ALSO in the 'stdin' field under the code window
type 2 things: an integer on the first line and some name on the second line. That would be your program input.

```
Scanner input = new Scanner(System.in);
        int age = input.nextInt();
        String name = input.next();
        System.out.println("Hi " + name + "you are "  + Integer.toString(age) + " year old " );
```

Look at the program output.

OK, now you are ready for the October POM:

1) 8 points
Write a Java program that takes 2 integers (int1 and int2) and a string (some_name) as an input and produces the following output:

Hey, some_name, everyone knows that int2 + int2 = sum

Where sum should be the sum of int1 and int2.

For example: For the following stdin:
125 50 Harry

Your program should print:
Hey, Harry, everyone knows that 125 + 50 = 175

2) 2 points:
Run your program with the following integers:

1000000000   2000000000

and try to figure out what is wrong.

Solution

1)
```
import java.util.Scanner;


public class Ideone
{
  public static void main (String[] args) throws java.lang.Exception
  {
        // your code goes here
        Scanner scan = new Scanner(System.in);

        int int1 = scan.nextInt();
        int int2 = scan.nextInt();
        String a_name = scan.next();

        int sum = int1 + int2;
        System.out.println("Hey, " + a_name + ", everyone knows that " + int1 + " + " + int2 + " = "
+ sum);

  }
}
```

2) The sum of 1000000000 and 2000000000 is bigger than the maximum value a variable of type int can store (2147483647).